FACULTAD ECONOMÍA Y NEGOCIOS
UNIVERSIDAD DE CHILE

DEPARTAMENTO DE ECONOMÍA

WITH ADDITIONAL ENFORCEMENT MECHANISMS, DOES COLLATERAL AVOID PONZI SCHEMES?

Autor: Thiago Revil y Juan Pablo Torres-Martínez.

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Thiago Revil   Juan Pablo Torres-Martínez

Abstract

In infinite horizon incomplete market economies, when the seizure of collateral guarantees is the only mechanism enforcing borrowers not to entirely default on their promises, equilibrium exists independently of the choice of collateral bundles.

In these economies, we analyze if generic additional enforcement mechanisms besides the seizure of collateral guarantees may eliminate the existence of physical feasible individuals' optimal plans. For this, we only need to focus on the decision problem of a price taker individual and on the effectiveness of the additional enforcement mechanisms, i.e. the amount of payments besides the value of collateral guarantees. Then, we show that there is a relationship between collateral requirements and the effectiveness of such additional mechanisms implying the non-existence of a solution for the individual's problem.

Keywords:

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KEYWORDS. Effective default enforcements, Collateral guarantees, Individual's optimality.
JEL classification: D50; D52.

1. INTRODUCTION

In modern financial markets, collateral guarantees play an important role in enforcing borrowers not to entirely default on their financial promises. These guarantees are used in several credit operations, from corporate bonds to collateralized mortgages obligations,[1] allowing markets to reduce credit risk and increase portfolio diversification. However, to protect investors from the excess of losses induced by large negative shocks in the value of collateral guarantees, financial markets may create and implement additional enforcement mechanisms against default. In this paper, we focus on the theoretical effects of this policy.

In general equilibrium models, the collateralization of financial contracts is mostly addressed when the only enforcement mechanism against default is the seizure of the associated collateral guarantees.[2] Particularly, in infinite horizon models with incomplete markets, Araujo, Páscoa and Torres-Martínez (2002) proved the existence of equilibrium independently of the choice of physical collateral guarantees, and without need to impose exogenous debt constraints or transversality

[1]That is, derivative assets secured by pools of individual mortgages, each of which is backed mostly by real estates.

[2]For a seminal two-period general equilibrium model of collateralized loans, see Geanakoplos and Zame (1992), and for an extension to more complex securitization structures see Steinert and Torres-Martínez (2007).

conditions. In their context, agents default only when the value of their debts is greater than that of the associated collateral requirements, otherwise they pay the original contracted amount. Thus, the net payoff of the joint operation of taking a loan and constituting the associated collateral is always non-negative. Therefore, by non-arbitrage, the value of any loan has to be less than the value of the respective collateral, precluding agents to get leveraged and eliminating Ponzi schemes.

However, Páscoa and Seghir (2007) have shown that the individual's problem may not have a physically feasible solution when linear utility penalties for default (of the form proposed by Shubik and Wilson (1977)) act as additional enforcement mechanism besides the seizure of collateral guarantees. They provide examples of deterministic economies where sufficiently harsh penalties induce agents to pay more than the depreciated collateral, which, by non-arbitrage, may lead the value of collateral requirements to be persistently lower than that of the loan. In such a context, given any budget feasible plan, agents may improve their utilities taking new loans, constituting the associated collateral guarantees, and using the remaining resources to increase their consumption. Therefore, it is impossible the existence of an optimal plan. Since the objective of the authors is to study equilibrium existence, they impose upper bounds on utility penalties such that their result is valid.

In this paper, we analyze if generic additional enforcement mechanisms besides the seizure of collateral guarantees may eliminate the existence of physical feasible individuals' optimal plans. For this, we only need to address the decision problem of a price taker agent in an economy analogous to that studied by Araujo, Páscoa and Torres-Martínez (2002). In such a context, we introduce *effective* additional enforcement mechanisms, i.e. mechanisms enforcing payments besides the value of the collateral guarantees. Then, we show that the individual's problem does not have a physically feasible solution when these additional mechanisms are persistently effective and collateral requirements are not large enough relative to the effectiveness of such mechanisms.

Regarding these additional mechanisms, we focus on their effectiveness on enforcing payments besides the value of the collateral requirements. Thus, we do not intend to explicitly model how markets impose these additional payments. However, with this *reduced form* approach, we can understand the causes generating the non-existence of a physical feasible solution for the individual's problem. Moreover, we show that such effectiveness play an important role in this analysis, since its relationship with collateral requirements is determinant for our result. Additionally, since agents perfectly foresee asset payments, there is no loss of generality in measuring effectiveness as a percentage of the remaining debt after the seizure of collateral requirements.

We may view our result from two different perspectives. From the first one, given a level of effectiveness of the additional mechanisms, we show that there are strictly positive upper bounds for collateral requirements under which agents have incentives to indefinitely postpone their debts through new credits, leading to the non-existence of an optimal utility maximizing plan. Therefore, the market choice of collateral guarantees becomes relevant. From the second one, we give theoretical foundations to Páscoa and Seghir (2007) examples. That is, given collateral requirements, we show that any sufficiently effective additional enforcement mechanism implies the non-existence of physical

feasible individuals' optimal plans. Hence, it is the effectiveness of these mechanisms that brings the main result, not any mechanism *per se*.

Additionally, there is a intuitive interpretation of our result. We analyze the behavior of price taker agents choosing state-contingent optimal plans in a context where there is no other friction in the markets besides a possible default on collateral backed financial securities. As we include enforcement mechanisms in addition to the seizure of collateral requirements, lenders may expect sufficiently large payments for their loans besides the value of the associated collateral requirements. In such a situation, these additional mechanisms, instead of strengthening, actually weakens the restrictions that collateral places on borrowing. In fact, lenders anticipate that, even in case of default, they still receive more than just the value of the collateral guarantees. Thus, by non-arbitrage, they lend more than the value of these guarantees. On the other side, borrowers have the incentive and the possibility to take new credits in order to pay their older ones, since there is no debt constraints or monitoring precluding agents to incur in a Ponzi scheme. Then, their behaviour leads to the non-existence of a solution for their utility maximization problem.

The remainder of the paper is organized as follows: Section 2 presents an infinite horizon economy with assets subject to default and with effective enforcement mechanisms in addition to collateral repossession. In Section 3 we show our main result. Some extensions are discussed in Sections 4 and 5.

2. Model

Consider a discrete-time infinite-horizon economy with uncertainty and symmetric information. Let S be the set of states of nature and $\mathbb{F}_t$ the information available at period $t \in T := \mathbb{N} \cup \{0\}$. $\mathbb{F}_t$ is a partition of S, and if $t' > t$, make $\mathbb{F}_{t'}$ finer than $\mathbb{F}_t$. Summarizing the uncertainty structure, define an event-tree as $D = \{(t, \sigma) \in T \times 2^S : t \in T, \sigma \in \mathbb{F}_t\}$, where a pair $\xi := (t, \sigma) \in D$ is called a node and $t(\xi) := t$ is the associated period of time. For simplicity, at $t = 0$ there is no information, that is $\mathbb{F}_0 := \{S\}$, and there is only one node, which is denoted by ξ_0.

A node $\xi' = (t', \psi')$ is a successor of $\xi = (t, \psi)$, denoted by $\xi' \geq \xi$, if $t' \geq t$ and $\psi' \subseteq \psi$. Given $\xi \in D$, the set of its successors is given by the subtree $D(\xi) := \{\mu \in D : \mu \geq \xi\}$. Also, for each $\xi \neq \xi_0$, since $\mathbb{F}_{t(\xi)}$ is finer than $\mathbb{F}_{t(\xi)-1}$, there is only one predecessor $\xi^- \in D$. We define ξ' as an immediate successor of ξ when it is in the set $\xi^+ := \{\xi' \in D : \xi' \geq \xi, t(\xi') = t(\xi) + 1\}$.

At each node ξ in the event-tree D there is a non-empty and finite set of commodities, L. These commodities may be traded in a competitive market at unitary prices $p_\xi = (p_{(\xi,l)})_{l \in L} \in \mathbb{R}_+^L$ by a non-empty set of consumers. Also, at any node $\xi > \xi_0$, there is a technology represented by a matrix with non-negative entries, $Y_\xi := ((Y_\xi)_{l,l'}; (l, l') \in L \times L)$, which transform commodity bundles consumed at ξ^-, and allows for durable commodities. Thus, for each $(l, l') \in L \times L$, $(Y_\xi)_{l,l'}$ is the amount of commodity l obtained at ξ if one unit of commodity l' is consumed at ξ^-. Also, let $W_\xi \in \mathbb{R}_+^L$ be the aggregate physical resources up to node ξ, while $W = (W_\xi)_{\xi \in D}$ is the plan of such resources.

There is a finite set of real assets $J(\xi)$ at each node $\xi \in D$. Each j in $J(\xi)$ is short-lived, has promises $A_{(\mu,j)} \in \mathbb{R}_{++}^L \cup \{0\}$ at $\mu \in \xi^+$, and is traded in competitive markets by a unitary price

$q_{(\xi,j)} \in \mathbb{R}_+$. Note that, when financial promises are non-trivial, its market value take into account all the commodities prices. This assumption may be intuitively understood as an indexation for asset payments using as a price index a referential bundle that may vary with the uncertainty of the economy. Thus, independently of prices, when at least a percentage of original promises is honored by borrowers, lenders maintain a minimal purchase power for every commodity.

Since assets are subject to credit risk, borrowers are burdened to constitute physical collateral guarantees in order to limit lenders' losses. Particularly, for every unit of an asset $j \in J(\xi)$ sold, borrowers must establish—and may consume—a bundle $C_{(\xi,j)} \in \mathbb{R}_+^L \setminus \{0\}$ that is seized by the market in case of default. For the sake of notation, let $J(D) := \{(\xi,j) \in D \times \cup_{\mu \in D} J(\mu) : j \in J(\xi)\}$ and $J^+(D) := \{(\mu,j) \in D \times \cup_{\eta \in D} J(\eta) : (\mu^-, j) \in J(D)\}$.

Furthermore, additional default enforcement mechanisms may exist. For each unit of asset $j \in J(\xi)$ sold, we let financial markets recover amounts of payments $(F_{(\mu,j)}(p_\mu))_{\mu \in \xi^+}$ that may be higher than the value of depreciated collateral guarantees in case of default. We allow generality in the type of additional enforcement mechanisms assuming that borrowers pay, and lenders expect to receive, a fixed percentage of the remaining debt, $\lambda_{(\mu,j)} \in [0,1]$. More formally, for every unit of asset $j \in J(\xi)$, each borrower pays at each $\mu \in \xi^+$ an amount

$$F_{(\mu,j)}(p_\mu) := \min\{p_\mu A_{(\mu,j)}, p_\mu Y_\mu C_{(\xi,j)}\} + \lambda_{(\mu,j)} \left[p_\mu A_{(\mu,j)} - p_\mu Y_\mu C_{(\xi,j)}\right]^+,$$

where $\lambda_{(\mu,j)} \in [0,1]$ is the *effectiveness* of additional enforcement mechanisms on asset j at node μ, and, for any $z \in \mathbb{R}$, $[z]^+ := \max\{z, 0\}$.

Our approach allows us to include in our analysis economic (i.e. those induced by legal contracts) and non-economic (e.g. moral sanctions, loss of reputations) default enforcement mechanisms, provided that these mechanisms may be summarized by a family of parameters of effectiveness, $(\lambda_{(\mu,j)})_{(\mu,j) \in J^+(D)}$. However, this last requirement do not induce loss of generality, since traders perfect foresee asset payments. In fact, we can always normalize financial payments as done above. Also, with this approach, it is possible to focus on the consequences of the effectiveness of such mechanisms on the individual's decision.

DEFINITION. *Given $(\mu,j) \in J^+(D)$, additional enforcement mechanisms are <u>effective</u> on asset j at a node μ when $(\lambda_{(\mu,j)}, A_{(\mu,j)})$ is a non-zero vector. Additional enforcement mechanisms are <u>persistently effective</u> in a subtree $D(\xi)$, if for any $\mu > \xi$, there is $j \in J(\mu^-)$ on which additional enforcement mechanisms are effective at μ.*

Definitions above not only depend on parameters $(\lambda_{(\mu,j)})_{(\mu,j) \in J^+(D)}$, but also on the non-triviality of the original promises. Thus, *effective additional enforcement mechanisms* means that, in the case of default, a strictly positive amount of resources is seized besides the depreciated collateral value.

In contrast to any equilibrium model, we focus in the non-existence of a physically feasible solution for the individual's problem. For these reason, it is sufficient to study a decision model where there is an infinitely lived agent, namely i, who perfectly foresees both market prices and the effectiveness of additional enforcement mechanisms.

Agent i has physical endowments $(w_\xi^i)_{\xi \in D} \in \mathbb{R}_+^{D \times L}$ and preferences represented by a utility function $U^i : \mathbb{R}_+^{D \times L} \to \mathbb{R}_+ \cup \{+\infty\}$. As commodities may be durable, we denote by W_ξ^i the cumulated endowments of agent i up to node ξ, which are recursively defined by: $W_\xi^i = w_\xi^i + Y_\xi W_{\xi^-}^i$, when $\xi > \xi_0$, and $W_{\xi_0}^i = w_{\xi_0}^i$, otherwise. Also, we assume that, for any $\xi \in D$, $W_\xi^i \leq W_\xi$.

Let $x_\xi \in \mathbb{R}_+^L$ be a bundle of autonomous consumption at node ξ (i.e. non-collateralized commodities). Also, define $\theta_{(\xi,j)}$ and $\varphi_{(\xi,j)}$ as the quantities of asset $j \in J(\xi)$ purchased and sold at the same node. Given $(p,q) \in \Pi := \mathbb{R}_+^{D \times L} \times \mathbb{R}_+^{J(D)}$, a plan

$$(x, \theta, \varphi) := \big((x_\xi, \theta_{(\xi,j)}, \varphi_{(\xi,j)}); \xi \in D, j \in J(\xi) \big) \in \mathbb{E} := \mathbb{R}_+^{D \times L} \times \mathbb{R}_+^{J(D)} \times \mathbb{R}_+^{J(D)}$$

is *budget feasible* for agent i at prices (p,q) when

$$(1) \qquad p_{\xi_0}(x_{\xi_0} - w_{\xi_0}^i) + p_{\xi_0} \sum_{j \in J(\xi_0)} C_{(\xi_0,j)} \varphi_{(\xi_0,j)} + \sum_{j \in J(\xi_0)} q_{(\xi_0,j)}(\theta_{(\xi_0,j)} - \varphi_{(\xi_0,j)}) \quad \leq \quad 0,$$

$$(2) \quad p_\xi(x_\xi - w_\xi^i) + p_\xi \sum_{j \in J(\xi)} C_{(\xi,j)} \varphi_{(\xi,j)} + \sum_{j \in J(\xi)} q_{(\xi,j)}(\theta_{(\xi,j)} - \varphi_{(\xi,j)})$$

$$\leq p_\xi Y_\xi x_{\xi^-} + \sum_{j \in J(\xi^-)} \big(p_\xi Y_\xi C_{(\xi^-,j)} \varphi_{(\xi^-,j)} + F_{(\xi,j)}(p_\xi)(\theta_{(\xi^-,j)} - \varphi_{(\xi^-,j)}) \big), \ \ \forall \xi > \xi_0.$$

Also, $(x, \theta, \varphi) \in \mathbb{E}$ is *physically feasible* if $x_\xi + \sum_{j \in J(\xi)} C_{(\xi,j)} \varphi_{(\xi,j)} \leq W_\xi$, for any $\xi \in D$. Finally, given $(p,q) \in \Pi$, the objective of agent i is to maximize the utility of his consumption, $U^i((x_\xi^i + \sum_{j \in J(\xi)} C_{(\xi,j)} \varphi_{(\xi,j)}^i)_{\xi \in D})$, choosing a budget feasible plan $(x^i, \theta^i, \varphi^i) \in \mathbb{E}$.

3. Enforcement mechanisms and the size of collateral bundles

In this section, we prove our main result: in contrast to the polar case studied by Araujo, Páscoa and Torres-Martínez (2002), the market choice of collateral bundles becomes relevant when there are *persistently effective* additional enforcement mechanisms besides collateral repossession. To achieve our objective, we impose the following hypotheses.

ASSUMPTION A1. *For any $\xi \in D$, $W_\xi^i \gg 0$.*

ASSUMPTION A2. *Given $z = (z_\xi) \in \mathbb{R}_+^{L \times D}$, define $U^i(z) = \sum_{\xi \in D} u_\xi^i(z_\xi)$, where for any $\xi \in D$, the function $u_\xi^i : \mathbb{R}_+^L \to \mathbb{R}_+$ is concave, continuous and strictly increasing. Also, $U^i(W)$ is finite.*[3]

Given $\eta \in D$, let $\Omega(\eta)$ be the set of assets $j \in J(\eta)$ on which additional enforcement mechanisms are effective at some node $\mu \in \eta^+$. Note that, given a subtree $D(\xi)$ in which additional enforcement mechanisms are persistently effective, $\Omega(\eta) \neq \emptyset$, $\forall \eta \in D(\xi)$.

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[3]Note that, as utilities are finite when evaluated in aggregate physical resources, the non-negativity of functions $u_\xi^i(\cdot)$ implies that, in any physical feasible allocation, agent's i utility is finite. Also, the concavity of the functions $(u_\xi^i)_{\xi \in D}$ implies that U^i is concave. Thus, for any $\sigma > 1$, $U^i(\sigma W)$ is also finite. In fact, $U^i(0.5W) < +\infty$ and, therefore, by concavity, $U^i(W) \geq \tau U^i(0.5W) + (1 - \tau)U^i(\sigma W)$, where $\tau = \frac{2\sigma - 2}{2\sigma - 1} \in (0,1)$, which implies that $U^i(\sigma W) < +\infty$.

THEOREM. *Under Assumptions A1-A2, suppose that additional enforcement mechanisms are persistently effective in a subtree $D(\xi)$. Independently of the prices $(p,q) \in \Pi$, there are strictly positive upper bounds $(\Psi_\eta)_{\eta \in D(\xi)}$ such that, if collateral bundles satisfy*

$$\min_{j \in \Omega(\eta)} \|C_{(\eta,j)}\|_\Sigma < \Psi_\eta, \qquad \forall \, \eta \in D(\xi),$$

then agent i's problem does not have a physically feasible solution.

PROOF. To shorten the notation, given $z = (z_1, \ldots, z_m) \in \mathbb{R}_+^m$, let $\|z\|_\Sigma := \sum_{s=1}^m z_s$ and $\|z\|_{\max} := \max_{1 \le s \le m} z_s$. Fix $\sigma > 1$. Given $\eta \ge \xi$, define

$$\overline{\pi}_\eta := \frac{U^i(W)}{\min_{l \in L} W_{(\eta,l)}^i}, \qquad \text{and} \qquad \underline{\pi}_\eta := \frac{u_\eta^i(\sigma W_\eta) - u_\eta^i(W_\eta)}{\sigma \|W_\eta\|_{\max}}.$$

Thus, for each $\eta \in D(\xi)$,

$$\Upsilon_\eta := \min_{j \in \Omega(\eta)} \sum_{\mu \in \eta^+} \lambda_{(\mu,j)} \underline{\pi}_\mu \min_{l \in L} A_{(\mu,j,l)}$$

is strictly positive, where $A_{(\mu,j,l)}$ denotes the l-th coordinate of $A_{(\mu,j)}$.

Suppose that, at each $\eta \in D(\xi)$,

$$\min_{j \in \Omega(\eta)} \|C_{(\eta,j)}\|_\Sigma < \Psi_\eta := \frac{\Upsilon_\eta}{\overline{\pi}_\eta}.$$

Assume that, for some $(p,q) \in \Pi$, there is an optimal budget and physically-feasible solution $(x^i, \theta^i, \varphi^i) \in \mathbb{E}$ for agent i's problem. It follows from Lemma 2 (see Appendix) that there are, for every $\eta \in D$, multipliers $\gamma_\eta^i \in \mathbb{R}_{++}$ and non-pecuniary returns (super-gradients) $v_\eta^i \in \partial u_\eta^i \left(x_\eta^i + \sum_{j \in J(\eta)} C_{(\eta,j)} \varphi_{(\eta,j)}^i \right)$ such that,[4] for each $j \in J(\eta)$,

$$\text{(3)} \qquad\qquad \gamma_\eta^i p_\eta \;\;\ge\;\; v_\eta^i + \sum_{\mu \in \eta^+} \gamma_\mu^i p_\mu Y_\mu,$$

$$\text{(4)} \qquad\qquad \gamma_\eta^i q_{(\eta,j)} \;\;\ge\;\; \sum_{\mu \in \eta^+} \gamma_\mu^i F_{(\mu,j)}(p_\mu).$$

Also, the family of multipliers $(\gamma_\eta^i)_{\eta \in D}$ can always be constructed to satisfy (see Lemma 2)

$$\text{(5)} \qquad \gamma_\eta^i p_\eta W_\eta^i \le \sum_{\eta \in D} u_\eta^i \left(x_\eta^i + \sum_{j \in J(\eta)} C_{(\eta,j)} \varphi_{(\eta,j)}^i \right) \le \sum_{\eta \in D} u_\eta^i(W_\eta),$$

where the last inequality follows from Assumption A2 jointly with the physical feasibility of agent i's consumption. Moreover, it is possible to find lower and upper bounds for $\gamma_\eta^i p_\eta$ at each $\eta \in D$. Assumption A1 and equation (5) ensure that $\gamma_\eta^i \|p_\eta\|_\Sigma \le \overline{\pi}_\eta$. Given $\eta \in D$, let $c_\eta^i = x_\eta^i + \sum_{j \in J(\eta)} C_{(\eta,j)} \varphi_\eta^i$ be the consumption bundle chosen by agent i at η. Using equation (3), we have that

$$\gamma_\eta^i p_\eta (\sigma W_\eta - c_\eta^i) \ge v_\eta^i (\sigma W_\eta - c_\eta^i) \ge u_\eta^i(\sigma W_\eta) - u_\eta^i(c_\eta^i) \ge u_\eta^i(\sigma W_\eta) - u_\eta^i(W_\eta) > 0.$$

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[4] Given a concave function $f : X \subset \mathbb{R}^n \to \mathbb{R} \cup \{-\infty\}$ and $x \in X$, the super-differential of f, $\partial f(x)$, is defined as the set of points $p \in X$, called super-gradients, such that $f(y) - f(x) \le p(y - x), \forall y \in X$.

Therefore, $\gamma_\eta^i \|p_\eta\|_\Sigma \geq \underline{\pi}_\eta$ and, at every node $\eta \in D(\xi)$, since $\Omega(\eta) \neq \emptyset$ and $\min_{j \in \Omega(\eta)} \|C_{(\eta,j)}\|_\Sigma < \Psi_\eta$, there exists $j \in \Omega(\eta)$ such that

$$(6) \qquad \gamma_\eta^i(p_\eta C_{(\eta,j)} - q_{(\eta,j)}) \quad \leq \quad \gamma_\eta^i p_\eta C_{(\eta,j)} - \sum_{\mu \in \eta^+} \gamma_\mu^i F_{(\mu,j)}(p_\mu) < 0,$$

where the last inequality follows from the definition of the upper bound of collateral requirements. Finally, using the Lemma 1 in the Appendix, we conclude that agent i's problem does not have a solution, contradicting the optimality of $(x^i, \theta^i, \varphi^i) \in \mathbb{E}$ under prices $(p, q) \in \Pi$. $\qquad \square$

Note that, by construction, upper bounds on collateral requirements, $(\Psi_\eta)_{\eta \in D(\xi)}$, depend only on the primitives of the economy and, for computational objectives, can be easily found.

Additionally, given collateral requirements, we can find lower bounds for the effectiveness on $D(\xi)$, $(\underline{\lambda}_\mu)_{\mu > \xi}$, such that, if $(\lambda_{(\mu,j)})_{\mu > \xi, j \in \Omega(\mu^-)}$ satisfy

$$\min_{j \in \Omega(\mu^-)} \lambda_{(\mu,j)} > \underline{\lambda}_\mu, \quad \forall \mu > \xi,$$

then, independently of prices, and for *any enforcement mechanism* inducing such effectiveness, there is no physical feasible solution for the agents' problem.[5] These lower bounds are informative, i.e. $\underline{\lambda}_\mu \in (0, 1)$, only for collateral requirements that are not high enough. In fact, for larger collateral requirements there is no default and, therefore, the market price of collateral requirements is always greater that the loan value.

4. A REMARK ON ENDOGENOUS EFFECTIVENESS

For each unit of asset sold in our model, we assume that the amount of payments besides the collateral guarantees is independent of the borrowers and does not depend on the history of default. This assumption allowed us to identify sold with purchased assets. Implicitly, we pool the debt contracts into derivatives following a trivial securitization, that is, by identifying prices and payments of debt markets with those of investment markets. However, our analysis may be extended for enforcement mechanisms in which the effectiveness is an endogenous and personalized variable.

For instance, consider a dynamic infinite horizon general equilibrium model in which, at any period, at any state of nature in such a period and for every agent, the access to credit markets and/or the available endowments depend on previous payments. Thus, in this new framework there may be endogenous incentives inducing borrowers to deliver payments larger than the depreciated value of collateral requirements. Suppose that financial markets still preserve some features from our original model. That is, each type of credit contract is securitized into only one derivative, primitive and derivative prices are equal, and lenders perfectly foresee the payments of derivatives. Specifically for this last feature, suppose that, in case of default, agents advance any payment in addition to

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[5]Using the same arguments in the proof of the Theorem, it is sufficient to take

$$\underline{\lambda}_\mu := \frac{\overline{\pi}_{\mu^-} \min_{j \in \Omega(\mu^-)} \|C_{(\mu^-,j)}\|_\Sigma}{\min_{j \in \Omega(\mu^-)} \sum_{\eta \in \mu} \underline{\pi}_\eta \min_{l \in L} A(\eta, j, l)}, \quad \forall \mu > \xi.$$

the depreciated collateral as a *percentage* of the remaining debt, facing payment functions with an analogous specification of our $(F_{(\mu,j)})_{(\mu,j) \in J^+(D)}$.

In this new context, under hypotheses on individual' characteristics analogous to Assumptions A1-A2, there are two conditions under which a result analogous to our Theorem holds:

(i) For any plan of prices, individual optimal allocations satisfy inequalities analogous to (3)-(5);

(ii) At the moment of the credit operation, borrowers are only required to constitute the associated collateral requirements.

In fact, assume that additional enforcement mechanisms are persistently effective in the event-tree or in a sub-event tree. If there is a physical feasible optimal allocation, using the same arguments of the proof of our theorem, condition (i) implies that if collateral requirements are not high enough, then unitary loan prices persistently exceed the associated collateral value. Thus, by condition (ii), the agent may improve his respective utility increasing borrowing along the event-tree. A contradiction.

Therefore, a natural question arises. When does an economy satisfy conditions (i) and (ii)? Regarding condition (i), it follows from Lemma 2 that any convex model satisfies it.[6] However, some enforcement mechanisms may induce non-convex budget sets. Even in these cases, (i) still holds if these non-convexities involve only the borrowers' deliveries.[7] On the other hand, condition (ii) holds unless there is some restriction on the short sales in addition to collateral requirements.

5. Extensions

Long-lived assets and endogenous collateral.

Our analysis also holds when long-lived real assets are available for trading. Essentially, non-arbitrage conditions associated to individual's problem are still valid (see Araujo, Páscoa and Torres-Martínez (2007)). On the other hand, if we want collateral requirements to become endogenous, as in Geanakoplos and Zame (2002), a pool of financial contracts can be offered at each node, with the same real promises but with different associated collateral bundles. Thus, the choice of financial contracts traded by borrowers induce an endogenous choice of the associated collateral. However, it is important to be careful with the size of the available collateral requirements, since individual's optimality may become incompatible with commodity market feasibility.

About the persistent effectiveness.

In our main result, we assume that additional enforcement mechanisms are effective in a subtree $D(\xi)$. However, it is possible to weaken this assumption, requiring only effectiveness in a infinite path along the event-tree. For this, we need some definitions.

Given $k \in \mathbb{N} \cup \{+\infty\}$, a *path of uncertainty* is a set $(\mu_n; n \in \mathbb{N}, n \le k) \subset D$ in which every μ_{n+1} is an immediate successor of μ_n, for each $n < k$. A set $B \subset D$ *does not have finite paths* when for any $k \in \mathbb{N}$ and for each path of uncertainty $(\mu_n; n \in \mathbb{N}, n \le k) \subset B$, there exists $\eta \in B$ such

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[6]We mean that a model is *convex* when agents' objective functions are concave and, for each vector of prices, budget sets are convex.

[7]Technically, in this case, the arguments in the proof of Lemma 2 can be remade by redefining the truncated problem $(P^{i,T})$ in such form that, for any $\eta \in D^T$, variables φ_η are fixed and equal to the optimal choices φ_η^i.

that $\eta \in \mu_k^+$. Additional enforcement mechanisms are *persistently effective in a path of uncertainty* Θ, if for any $\mu \in \Theta$, there is $j \in J(\mu^-)$ on which additional enforcement mechanisms are effective at μ. For any path of uncertainty $\Theta := (\mu_n; n \in \mathbb{N})$ in which additional enforcement mechanisms are persistently effective, define $\text{Eff}(\Theta) \subset D(\mu_1)$ as the maximal connected set containing Θ and having, at each $\mu \in \text{Eff}(\Theta)$, at least one $j \in J(\mu^-)$ on which additional enforcement mechanisms are effective at μ.[8]

Note that, given a path of uncertainty Θ, in which additional enforcement mechanisms are persistently effective, if $\text{Eff}(\Theta)$ does not have finite paths, then $\Omega(\eta) \neq \emptyset$, $\forall \eta \in \text{Eff}(\Theta)$. Thus, Lemma 1 holds when additional enforcement mechanisms are persistently effective *only* in a path of uncertainty Θ, provided that $\text{Eff}(\Theta)$ does not have finite paths. In fact, if for any $\eta \in \text{Eff}(\Theta)$, there exists $j \in J(\eta)$ for which $p_\eta C_{(\eta,j)} - q_{(\eta,j)} < 0$, then agent i may increases his borrowing at the first node of Θ paying his future commitments *either* using new credit at the nodes in which there is effectiveness or delivering depreciated collateral guarantees for the nodes $\mu \notin \text{Eff}(\Theta)$ such that $\mu^- \in \text{Eff}(\Theta)$.

Therefore, it follows that our main theorem holds when additional enforcement mechanisms are persistently effective in a path of uncertainty Θ for which $\text{Eff}(\Theta)$ does not have finite paths.

APPENDIX

In a context of collateralized assets and *linear utility penalties for default*, Páscoa and Seghir (2007) show that Ponzi schemes could be implemented if there exists a subtree $D(\xi)$ such that, for every node $\mu \geq \xi$, there is always some asset $j \in J(\mu)$ whose price exceeds the respective collateral value, $p_\mu C_{(\mu,j)} - q_{(\mu,j)} < 0$ (see Remark 3.1 in Páscoa and Seghir (2007)). In such event, the individual's problem does not have a finite solution. In our context, the same result follows by analogous arguments.

LEMMA 1. *Assume that, given $x \in \mathbb{R}_+^{L \times D}$, if $U^i(x)$ is finite, then $U^i(y) > U^i(x)$ for any $y > x$. Also, suppose that additional enforcement mechanisms are persistently effective in a subtree $D(\xi)$ such that, for any $\eta \in D(\xi)$, there exists $j \in J(\eta)$ for which $p_\eta C_{(\eta,j)} - q_{(\eta,j)} < 0$. Then, agent i's individual problem does not have a finite solution, otherwise, Ponzi schemes could be implemented.*

PROOF. Assume there is a budget feasible plan for agent i, $(x^i, \theta^i, \varphi^i)$, that gives a finite optimum. Under the monotonicity condition stated in the Lemma, $p_\eta \gg 0$ for every node $\eta \in D(\xi)$. For each $\eta \in D(\xi)$, let $J^1(\eta) = \{j \in J(\eta) : p_\eta C_{(\eta,j)} - q_{(\eta,j)} < 0\}$. Now, consider the allocation $(x_\xi, \theta_\xi, \varphi_\xi)_{\xi \in D}$, with

$$\left((x_\mu, \theta_\mu, \varphi_\mu); (\theta_\eta, \varphi_{(\eta,j)})\right)_{\mu \notin D(\xi),\, \eta \in D(\xi)} = \left((x_\mu^i, \theta_\mu^i, \varphi_\mu^i); (\theta_\eta^i, \varphi_{(\eta,j)}^i)\right)_{\mu \notin D(\xi),\, \eta \in D(\xi)}, \quad \forall j \in J(\eta) \backslash J^1(\eta)$$

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[8]A set $B \subset D$ is connected when, for each pair $(\xi, \mu) \in B \times B$, such that $\mu \geq \xi$, the only path of uncertainty connecting ξ to μ is contained in B. Given $\xi \in D$, a set $B \subset D(\xi)$ is maximal relative to a property A when there is no other subset of $D(\xi)$ containing itself and satisfying A.

and

$$\varphi_{(\eta,j)} \quad = \quad \varphi^i_{(\eta,j)} + \delta_\eta, \quad \forall \eta \in D(\xi), \forall j \in J^1(\eta),$$

$$x_{(\eta,l)} \quad = \quad x^i_{(\eta,l)} + \frac{1}{(\#L)\,p_{(\eta,l)}} \sum_{j \in J^1(\eta)} \left(q_{(\eta,j)} - p_\eta C_{(\eta,j)} \right) \delta_\eta, \quad \forall l \in L, \text{ if the node } \eta = \xi,$$

$$x_{(\eta,l)} \quad = \quad x^i_{(\eta,l)} + \frac{1}{(\#L)\,p_{(\eta,l)}} \sum_{j \in J^1(\eta)} \left(q_{(\eta,j)} - p_\eta C_{(\eta,j)} \right) \delta_\eta$$

$$- \frac{1}{(\#L)\,p_{(\eta,l)}} \sum_{j \in J^1(\eta^-)} p_\eta A_{(\eta,j)}\, \delta_{\eta^-}, \ \forall \eta > \xi, \forall l \in L,$$

where the plan $(\delta_\eta)_{\eta \in D(\xi)}$ is chosen in such form that the following conditions hold,

$$(7) \qquad\qquad \sum_{j \in J^1(\xi)} \left(q_{(\xi,j)} - p_\xi C_{(\xi,j)} \right) \delta_\xi \quad > \quad 0,$$

$$(8) \qquad\qquad \sum_{j \in J^1(\eta)} \left(q_{(\eta,j)} - p_\eta C_{(\eta,j)} \right) \delta_\eta \quad > \quad \sum_{j \in J^1(\eta^-)} p_\eta A_{(\eta,j)}\, \delta_{\eta^-}, \quad \forall \eta > \xi.$$

It follows that $(x_\xi, \theta_\xi, \varphi_\xi)_{\xi \in D}$ is budget feasible at prices (p,q). Moreover, equations above show that Ponzi schemes are possible at prices (p,q). In fact, agent i increases his borrowing at ξ and pays his future commitments by using new credit. It follows that $(x_\xi, \theta_\xi, \varphi_\xi)_{\xi \in D}$ improves the utility level of agent i, contradicting the optimality of $(x^i, \theta^i, \varphi^i)$. $\qquad\square$

The following result and its demonstration are analogous to Proposition 1 in Araujo, Páscoa and Torres-Martínez (2007). However, as slight modifications are necessary we present the whole proof for the readers.

LEMMA 2. *Let $(p,q) \in \Pi$ and fix a budget and physically feasible plan $z^i := (x^i, \theta^i, \varphi^i) \in \mathbb{E}$. Under Assumptions A1 and A2, if z^i is an optimal allocation for agent i's problem at prices (p,q), then for every $\eta \in D$, the function u^i_η is super-differentiable at the point $c^i_\eta := x^i_\eta + \sum_{j \in J(\eta)} C_{(\eta,j)} \varphi^i_{(\eta,j)}$, there are multipliers $\gamma^i_\eta \in \mathbb{R}_{++}$ and super-gradients $v^i_\eta \in \partial u^i_\eta \left(c^i_\eta \right)$ such that, for each $j \in J(\eta)$,*

$$(9) \qquad\qquad \gamma^i_\eta p_\eta \quad \geq \quad v^i_\eta + \sum_{\mu \in \eta^+} \gamma^i_\mu p_\mu Y_\mu,$$

$$(10) \qquad\qquad \gamma^i_\eta q_{(\eta,j)} \quad \geq \quad \sum_{\mu \in \eta^+} \gamma^i_\mu F_{(\mu,j)}(p_\mu).$$

Also, the plan of multipliers $(\gamma^i_\eta)_{\eta \in D}$ satisfy

$$(11) \qquad\qquad \gamma^i_\eta p_\eta W^i_\eta \leq \sum_{\eta \in D} u^i_\eta(c^i_\eta).$$

PROOF. Given $T \in \mathbb{N}$, define $D_T = \{\eta \in D : t(\eta) = T\}$ and $D^T = \{\eta \in D : \eta \in \bigcup_{k=0}^T D_k\}$. For any $\eta \in D$, let $Z(\eta) = \mathbb{R}^L_+ \times \mathbb{R}^{J(\eta)}_+ \times \mathbb{R}^{J(\eta)}_+$. For convenience of notations, let $z_{\xi_0^-} := 0 \in Z(\xi_0^-)$, where

$Z(\xi_0^-) := \mathbb{R}_+^L$. Consider the optimization problem:

$$(P^{i,T}) \quad \begin{array}{c} \max\limits_{\eta \in D^T} \sum u_\eta^i \left(x_\eta + \sum_{j \in J(\eta)} C_{(\eta,j)} \varphi_{(\eta,j)} \right) \\ \text{s.t.} \end{array} \begin{cases} z_\eta := (x_\eta, \theta_\eta, \varphi_\eta) \in Z(\eta) & \forall \eta \in D^T, \\ g_\eta^i(z_\eta, z_{\eta^-}; p, q) & \leq \ 0, \quad \forall \eta \in D^T, \\ x_\eta + \sum_{j \in J(\eta)} C_{(\eta,j)} \varphi_{(\eta,j)} & \leq \ 2W_\eta, \ \forall \eta \in D^T, \\ z_\eta & \in \ [0, z_\eta^i], \ \forall \eta \in D_T. \end{cases}$$

where the inequality $g_\eta^i(z_\eta, z_{\eta^-}; p, q) \leq 0$ represents the budget constraint at node η, that is, inequality (1) or (2), and given $(x, y) \in \mathbb{R}^m \times \mathbb{R}^m$, the interval $[x, y] := \{ z \in \mathbb{R}^m : \exists a \in [0, 1], z = ax + (1-a)y \}$. It follows from the existence of an optimal individual plan at prices (p, q) that there exists a solution for $(P^{i,T})$, namely $(z_\eta^{i,T})_{\eta \in D^T}$.[9]

Given $\eta \in D$, define the concave function $\nu_\eta^i : \mathbb{R}^L \times \mathbb{R}^{J(\eta)} \times \mathbb{R}^{J(\eta)} \to \mathbb{R} \cup \{-\infty\}$ as

$$\nu_\eta^i(z_\eta) = \begin{cases} u_\eta^i \left(x_\eta + \sum_{j \in J(\eta)} C_{(\eta,j)} \varphi_{(\eta,j)} \right) & \text{if } x_\eta + \sum_{j \in J(\eta)} C_{(\eta,j)} \varphi_{(\eta,j)} \geq 0; \\ -\infty & \text{otherwise.} \end{cases} \quad ,$$

where $z_\eta = (x_\eta, \theta_\eta, \varphi_\eta)$. It follows that, for any $T \geq 1$, $\sum_{\eta \in D^T} \nu_\eta^i(z_\eta^{i,T}) \leq \sum_{\eta \in D} \nu_\eta^i(z_\eta^i)$.[10]

For each $\eta \in D$ and $\gamma_\eta \in \mathbb{R}_+$, define $\mathcal{L}_\eta^i(\cdot, \gamma; p, q) : \mathbb{Z}(\eta) \times \mathbb{Z}(\eta^-) \to \mathbb{R}$ as

$$\mathcal{L}_\eta^i(z_\eta, z_{\eta^-}, \gamma_\eta; p, q) = \nu_\eta^i(z_\eta) - \gamma_\eta \, g_\eta^i(z_\eta, z_{\eta^-}; p, q).$$

Given $T \in \mathbb{N}$, for each $\eta \in D^{T-1}$, define the set $\Xi^T(\eta)$ as the family of allocations $(x_\eta, \theta_\eta, \varphi_\eta) \in Z(\eta)$ that satisfies $x_\eta + \sum_{j \in J(\eta)} C_{(\eta,j)} \varphi_{(\eta,j)} \leq 2W_\eta$. Also, for any $\eta \in D_T$, let $\Xi^T(\eta)$ be the set of allocations $(x_\eta, \theta_\eta, \varphi_\eta) \in Z(\eta)$ that satisfies both $x_\eta + \sum_{j \in J(\eta)} C_{(\eta,j)} \varphi_{(\eta,j)} \leq 2W_\eta$ and $(x_\eta, \theta_\eta, \varphi_\eta) \in [0, z_\eta^i]$. Let $\Xi^T := \prod_{\eta \in D^T} \Xi^T(\eta)$.

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[9]In fact, define a new problem $(\tilde{P}^{i,T})$,

$$(\tilde{P}^{i,T}) \quad \begin{array}{c} \max\limits_{\eta \in D^T} \sum u_\eta^i \left(x_\eta + \sum_{j \in J(\eta)} C_{(\eta,j)} \varphi_{(\eta,j)} \right) \\ \text{s.t.} \end{array} \begin{cases} z_\eta := (x_\eta, \theta_\eta, \varphi_\eta) \in Z(\eta) & \forall \eta \in D^T, \\ g_\eta^i(z_\eta, z_{\eta^-}; p, q) & \leq \ 0, \quad \forall \eta \in D^T, \\ x_\eta + \sum_{j \in J(\eta)} C_{(\eta,j)} \varphi_{(\eta,j)} & \leq \ 2W_\eta, \ \forall \eta \in D^T, \\ z_\eta & \in \ [0, z_\eta^i], \ \forall \eta \in D_T, \\ \text{If} \ q_{(\eta,j)} = 0 & \text{then} \ \theta_{(\eta,j)} = 0. \end{cases}$$

Under Assumption A2 the objective function on $(\tilde{P}^{i,T})$ is continuous, and the set of admissible allocations is compact in $\prod_{\eta \in D^T} Z(\eta)$. Note that, to ensure this it is necessary to have non-zero collateral requirements, otherwise, long and short positions are unbounded.

Thus, there is a solution $(z_\eta^{i,T})_{\eta \in D^T}$. Moreover, this solution for $(\tilde{P}^{i,T})$ is also an optimal choice for $(P^{i,T})$. Essentially, the existence of a *finite* optimum at prices (p, q) for the agent i's problem ensure that, when $q_{(\eta,j)} = 0$, the payments $F_{(\mu,j)}(p_\mu)$ must be equal zero, for each $\mu \in \eta^+$. Thus, when $q_{(\eta,j)} = 0$, choosing positives amounts of $\theta_{(\eta,j)}$ does not induce any gains.

[10]Note that, otherwise, agent i improve his utility in D choosing the allocation $(z_\eta^{i,T})_{\eta \in D^T}$ in the sub-tree D^T, without making any (physical or financial) trade after the nodes with date T.

It follows from Rockafellar (1997, Theorem 28.3), that there exist non-negative multipliers $(\gamma_\eta^{i,T})_{\eta \in D^T}$ such that the following saddle point property holds,

$$(12) \qquad \sum_{\eta \in D^T} \mathcal{L}_\eta^i(z_\eta, z_{\eta_-}, \gamma_\eta^{i,T}; p, q) \leq \sum_{\eta \in D^T} \mathcal{L}_\eta^i(z_\eta^{i,T}, z_{\eta_-}^{i,T}, \gamma_\eta^{i,T}; p, q), \quad \forall (z_\eta)_{\eta \in D^T} \in \Xi^T,$$

and $\gamma_\eta^{i,T} g_\eta^i(z_\eta^{i,T}, z_{\eta_-}^{i,T}; p, q) = 0$.

Claim A. *For each $\mu \in D$, the sequence $(\gamma_\mu^{i,T})_{T \geq t(\mu)}$ is bounded. Moreover, given $T > t(\mu)$,*

$$\nu_\mu^i(a_\mu) - \nu_\mu^i(z_\mu^i) \leq \left(\gamma_\mu^{i,T} \nabla_1 g_\mu^i(p, q) + \sum_{\eta \in \mu^+} \gamma_\eta^{i,T} \nabla_2 g_\eta^i(p, q) \right) \cdot (a_\eta - z_\eta^i) + \sum_{\xi \in D \setminus D^{T-1}} \nu_\xi^i(z_\xi^i), \quad \forall a_\mu \in \Xi^T(\mu),$$

where, for any $\eta \in D$, the vector $(\nabla_1 g_\eta^i(p, q), \nabla_2 g_\eta^i(p, q))$ is defined by

$$\begin{aligned}
\nabla_1 g_\eta^i(p, q) &= \left(p_\eta, q_\eta, (p_\eta C_{(\eta,j)} - q_{(\eta,j)})_{j \in J(\eta)} \right), \\
\nabla_2 g_\eta^i(p, q) &= \left(-p_\eta Y_\eta, (F_{(\eta,j)})_{j \in J(\eta)}, (p_\eta Y_\eta C_{(\eta,j)} - F_{(\eta,j)})_{j \in J(\eta)} \right).
\end{aligned}$$

Proof. Given $t \leq T$, substitute the following allocation in inequality (12)

$$z_\eta = \begin{cases} (W_\eta^i, 0, 0), & \forall \, \eta \in D^{t-1}, \\ (0, 0, 0), & \forall \eta \in D^T \setminus D^{t-1}. \end{cases}$$

We have:

$$(13) \qquad \sum_{\eta \in D_t} \gamma_\eta^{i,T} p_\eta W_\eta^i \leq \sum_{\eta \in D^T} \nu_\eta^i(z_\eta^{i,T}) \leq \sum_{\eta \in D} \nu_\eta^i(z_\eta^i).$$

Assumptions A1 ensure that, for each $\eta \in D$, $\min_{l \in L} W_{(\eta,l)}^i > 0$. Also, Assumption A2 implies that $\|p_\eta\|_\Sigma > 0$, guaranteing that, for each $\mu \in D$, the sequence $(\gamma_\mu^{i,T})_{T > t(\mu)}$ is bounded.

On the other hand, given $(z_\eta)_{\eta \in D^T} \in \Xi^T$, using (12), we have that

$$\sum_{\eta \in D^T} \mathcal{L}_\eta^i(z_\eta, z_{\eta_-}, \gamma_\eta^{i,T}; p, q) \leq \sum_{\eta \in D} \nu_\eta^i(z_\eta^i).$$

Thus, fix $\mu \in D^{T-1}$ and $a_\mu \in \Xi^T(\mu)$. If we evaluate inequality above in

$$z_\eta = \begin{cases} z_\eta^i, & \forall \, \eta \neq \mu, \\ a_\mu, & \text{for } \eta = \mu, \end{cases}$$

we obtain

$$(14) \quad \nu_\mu^i(a_\mu) - \gamma_\mu^{i,T} g_\mu^i(a_\mu, z_{\mu_-}^i; p, q) - \sum_{\eta \in \mu^+} \gamma_\eta^{i,T} g_\eta^i(z_\eta^i, a_\mu; p, q) \leq \nu_\mu^i(z_\mu^i) + \sum_{\eta \in D \setminus D^T} \nu_\eta^i(z_\eta^i).$$

Since functions $(g_\xi^i(\cdot; p, q); \xi \in D)$ are affine, we have

$$\begin{aligned}
g_\mu^i(a_\mu, z_{\mu_-}^i; p, q) &= \nabla_1 g_\mu^i(p, q) \cdot a_\mu - p_\mu \omega_\mu^i + \nabla_2 g_\mu^i(p, q) \cdot z_{\mu_-}^i \\
g_\eta^i(z_\eta^i, a_\mu; p, q) &= \nabla_1 g_\eta^i(p, q) \cdot z_\eta^i - p_\eta \omega_\eta^i + \nabla_2 g_\eta^i(p, q) \cdot a_\mu, \quad \forall \eta \in \mu^+.
\end{aligned}$$

Also, budget feasibility of $(z^i_\eta)_{\eta \in D}$ at prices (p, q), jointly with monotonicity of preferences, ensure that,

$$-p_\mu \omega^i_\mu + \nabla_2 g^i_\mu(p, q) \cdot z^i_{\mu-} = -\nabla_1 g^i_\mu(p, q) \cdot z^i_\mu,$$
$$\nabla_1 g^i_\eta(p, q) \cdot z^i_\eta - p_\eta \omega^i_\eta = -\nabla_2 g^i_\eta(p, q) \cdot z^i_\mu, \quad \forall \eta \in \mu^+.$$

Therefore,

$$\gamma^{i,T}_\mu g^i_\mu(a_\mu, z^i_{\mu-}; p, q) + \sum_{\eta \in \mu^+} \gamma^{i,T}_\eta g^i_\eta(z^i_\eta, a_\mu; p, q) = \left(\gamma^{i,T}_\mu \nabla_1 g^i_\mu(p, q) + \sum_{\eta \in \mu^+} \gamma^{i,T}_\eta \nabla_2 g^i_\eta(p, q) \right) \cdot (a_\mu - z^i_\mu).$$

Using (14), we conclude the proof. $\boxtimes$

Since D is countable and, for any node η, the sequence $(\gamma^{i,T}_\eta)_{T \geq t(\eta)}$ is bounded, using Tychonoff Theorem (see Aliprantis and Border (1999, Theorem 2.57)), there is a common subsequence $(T_k)_{k \in \mathbb{N}} \subset \mathbb{N}$ and non-negative multipliers, $(\gamma^i_\eta)_{\eta \in D}$, such that, for each $\eta \in D$, $\lim_{k \to \infty} \gamma^{i,T_k}_\eta = \gamma^i_\eta$ and $\gamma^i_\eta g^i_\eta(z^i_\eta, z^i_{\eta-}; p, q) = 0$, where, as we said above, the last equation follows from the strictly monotonicity of u^i_η. Moreover, taking the limit as T goes to infinity in inequality (13) we obtain that

$$(15) \qquad \sum_{\eta \in D_t} \gamma^i_\eta p_\eta W^i_\eta \leq \sum_{\eta \in D} \nu^i_\eta(z^i_\eta), \quad \forall t \geq 0.$$

Therefore, equation (11) follows.

Since for any $\eta \in D$, $\Xi^{s_1}(\eta) = \Xi^{s_2}(\eta)$ when $\min\{s_1, s_2\} > t(\eta)$, it follows from the inequality in the statement of Claim above, taking the limit as T goes to infinity, that

$$(16) \nu^i_\eta(a_\eta) - \nu^i_\eta(z^i_\eta) \leq \left(\gamma^i_\eta \nabla_1 g^i_\eta(p, q) + \sum_{\mu \in \eta^+} \gamma^i_\mu \nabla_2 g^i_\mu(p, q) \right) \cdot (a_\eta - z^i_\eta), \quad \forall a_\eta \in \Xi^{t(\eta)+1}(\eta).$$

Thus,

$$\left(\gamma^i_\eta \nabla_1 g^i_\eta(p, q) + \sum_{\mu \in \eta^+} \gamma^i_\mu \nabla_2 g^i_\mu(p, q) \right) \in \partial \left(\nu^i_\eta + \delta_{Z_1(\eta)} + \delta_{Z_2(\eta)} \right)(z^i_\eta),$$

where the functions $\delta_{Z_h(\eta)} : \mathbb{R}^L \times \mathbb{R}^{J(\eta)} \times \mathbb{R}^{J(\eta)} \to \mathbb{R} \cup \{-\infty\}$, $h \in \{1, 2\}$, satisfy

$$\delta_{Z_1(\eta)}(x_\eta, \theta_\eta, \varphi_\eta) = \begin{cases} 0, & \text{if } (x_\eta, \theta_\eta, \varphi_\eta) \in Z(\eta), \\ -\infty, & \text{otherwise.} \end{cases}$$

$$\delta_{Z_2(\eta)}(x_\eta, \theta_\eta, \varphi_\eta) = \begin{cases} 0, & \text{if } x_\eta + \sum_{j \in J(\eta)} C_{(\eta,j)} \varphi_{(\eta,j)} \leq 2W_\eta, \\ -\infty, & \text{otherwise.} \end{cases}$$

where $z_\eta = (x_\eta, \theta_\eta, \varphi_\eta) \in \mathbb{R}^L \times \mathbb{R}^{J(\eta)} \times \mathbb{R}^{J(\eta)}$. Since the plan $(z^i_\eta)_{\eta \in D}$ is physically feasible, there exists a neighborhood V of z^i_η such that $\delta_{Z_2(\eta)}(b) = 0$ for every $b \in V$. Then, we have that $\partial \delta_{Z_2(\eta)}(z^i_\eta) = \{0\}$. Also, it follows by Theorem 23.8 and 23.9 in Rockafellar (1997), that there exists $v^i_\eta \in \partial u^i_\eta(c^i_\eta)$ and $\kappa^i_\eta \in \partial \delta_{Z(\eta)}(x^i_\eta, \theta^i_\eta, \varphi^i_\eta)$ such that

$$(17) \qquad \gamma^i_\eta \nabla_1 g^i_\eta(p, q) + \sum_{\mu \in \eta^+} \gamma^i_\mu \nabla_2 g^i_\mu(p, q) = (v^i_\eta, 0, (C_{(\eta,j)} v^i_\eta)_{j \in J(\eta)}) + \kappa^i_\eta.$$

Notice that, by definition, for each $z_\eta \geq 0, \kappa \in \partial \delta_{Z(\eta)}(z_\eta) \Leftrightarrow 0 \leq \kappa(y - z_\eta), \ \forall \ y \geq 0$, therefore, $\kappa_\eta^i \geq 0$. Thus, the inequalities stated in the lemma hold from equation (17). On the other hand, strictly monotonicity of function u_η^i, ensure that $v_\eta^i \gg 0$ and, therefore, it follows from (9), that γ_η^i is *strictly positive*. $\qquad\qquad\square$

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Department of Economics, Pontifical Catholic University of Rio de Janeiro (PUC-Rio)

Rua Marquês de São Vicente 225, Gávea, 22453-900 Rio de Janeiro, Brazil.

E-mail address: revilth@econ.puc-rio.br

Department of Economics, University of Chile

Diagonal Paraguay 257 office 1504, Santiago, Chile.

E-mail address: jptorres.martinez@gmail.com