Nota técnica

# THE EXISTENCE OF COMPENSATED EQUILIBRIUM

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## Introduction

The purpose of this note is twofold. First and foremost, we wish to make Arrow and Hahn's proof of the existence of compensated equilibrium a little less mysterious by expanding and elucidating upon the various interconnecting steps presented in the text;[1] and then we wish to present proofs of some important theorems in the realm of topology which the authors use and which, I presume, the general reader does not find so obvious. In particular, we shall prove that: (1) Any closed subspace of a compact space is compact; (2) Any continuous image of a compact space is compact and will discuss briefly Tychonoff's theorem: the product of any non empty class of compact spaces is compact. So as to not interfere with our principal task these proofs shall be presented in the Appendix, for the benefit of the inquisitive reader.

## The Central Ideas

We start out by defining what we mean by a "compensated equilibrium".

### Definition I

A price vector $p^*$, a utility allocation $u^*$, a consumption allocation $x^*$, and a production allocation $y^*$, constitute a "compensated equilibrium" if:

(a) $p^* > 0$;

(b) $\sum_h x_h^* \leq \sum_f y_f^* + \sum_h \bar{x}_h$;

(c) $y_f^*$ maximizes $p^* y_f$ subject to $y_f \in Y_f$;

(d) $x_h^*$ minimizes $p^* x_h$ subject to $U_h(x_h) \geq u_h^*$;

(e) $p^* x_h^* = M_h^*$

The essential distinction between a competitive equilibrium (whose existence is the theme of the next section in the text) and a compensated one, is that while in the first case we wish to maximize utility subject to a budget constraint, in the latter we set out to minimize expenditures mindful that our particular household's utility does not fall below a certain preassigned level.

### Definition II

We define the budgetary surplus for househod h: $s_h(p,w)$, as the difference between that household's income and its total expenditures, given p and a feasible allocation w:

$$s_h(p, w) = p\bar{x}_h + \sum_f d_{hf}(py_f) - px_h = M_h(p, y) - px_h$$

[1] Arrow, K. y Hahn F. (1971), *General Competitive Analysis*, San Francisco, Holden–Day.

Note: The definition giveu in the text for $s_h$ (p, w) has a misprint. The summation should not be multiplied by the vertor p.

**Definition III**

The set of "relative utility vectors" is defined by:

$$S_H = \{v \mid v \geq 0, \ \Sigma v_h = 1\}$$

**Definition IV**

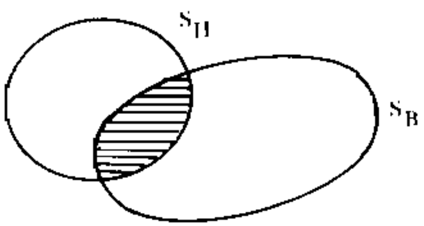
The set of "puuishable households" is defined by:

$$S_B = \{v \mid v_h = 0 \ \text{if} \ s_h(p, w) < 0\}$$

This latter set fixes at zero the relative utility of a household if its expenses exceed its income. Lemma 5.3 assures us that these household's utility will in fact be set equal to zero.

**Definition V**

$$V(p, W) = S_H \cap S_B$$

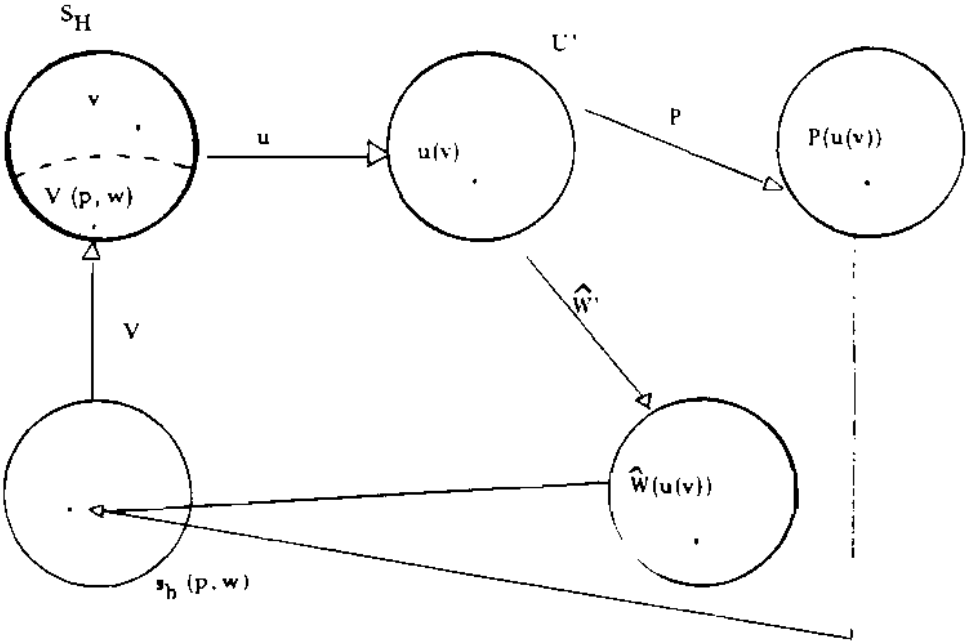



Consider a price vector p, a relative utility allocation v and a feasible commodity allocation w; not necessarily consistent with each other. Let us focus, for the moment, our attention on v, the relative utility allocation. In lemma 5.3 it has been proved that there exists a continuous function u which maps v ($\in S_H$) into the set of oonnegative Pareto efficient points also known as the Pareto frontier: U. We know; however, from theorem 4.4 that if u(v) is Pareto efficient there exists a price vector $p > 0$ for which the value of excess demand pz is nonnegative and which, essencially, given a preassigned level of utility, minimizes the household's expenditures, maximizes the firms's profits and satisfies the social budget constraint. In few words, there exists a price vector which supports the Pareto efficient allocation u(v). Call this price vector P(u).

Let us consider now the feasible commodity allocation w. Since u is Pareto efficient (i.e. feasible and not dominated by any other feasible utility allocation), surely there exists a feasible allocation, $\hat{W}(u)$ which does not dominate our original u.

A price vector P(u) and a feasible allocation $\dot{W}(u)$ thus found define immediately budgetary surpluses and consequently a new set of relative utilities V (p, w).

We can represent the above ideas by means of the following diagram




Note that the mapping so constructed lends itself very nicely for the use of some sort of fixed point argument. That is, we start with a relative utility allocation and, as the above arguments show, we map back to the set of relative utility allocations. We will show that there exists at least one point in the set of relative utility allocations which, together with a vector of prices and a feasible commodity allocation, are mapped back into themselves by the above correspondence.

**Definition VI**

We define the k – 1 dimensional unit simplex:

$$S^{k-1} = \{p \text{ in } R^k \quad \sum_{i=1}^{k} p_i = 1\}$$

**Lemma I**

A k – 1 dimensional unit simplex is a compact convex set.

Pf:

Let $x_1, x_2, \ldots, x_i, \ldots x_k$, be $\geqslant 0$ for all i and $\sum_{i=1}^{k} x_i = 1$

Let $x^1 = (x_1^1, \ldots, x_k^1)$ and $x^2 = (x_i^2, \ldots x_k^2)$ be two elements in the k–1 dimensional simplex. Then, for $\lambda \epsilon (0,1)$,

$$\sum_{i=1}^{k} [\lambda x_i^1 + (1-\lambda) x_i^2] = \lambda \sum_{i=1}^{k} x_i^1 + (1-\lambda) \sum_{i=1}^{k} x_i^2 = \lambda + (1-\lambda) = 1$$

So the k—1 dimensional unit simplex is indeed convex.

By construction it is clearly bounded since, in fact, any ponit on the simplex is no more than one unit away from the origin. Closure of the simplex follows if we observe that points on it do not get arbitrarily close to any point outside of it. Hence compactness.

**Important Observation.** The compactness of $S_H$ cau be established iu a different manner. In the previous section it was shown that v(u) maps U' (the Pareto Frontier) into the unit simplex $S_H$. Furthermore v(u) is continuous and the Pareto frontier is compact. Since continuous images of compact sets are compact, it follows that $S_H$ is compact. This theorem, which was used by the authors to show that the set of feasible utility allocations is compact is proved in the Appendix (See Theorem II).

The domain on which our correspondence P(u(v))X V(p, w)X W(u(v)) is defined in $s_h$ X $S_H$ X $\hat{W}$ which is the cross product of two compact convex sets with $\hat{W}$ which, by theorem 4.2, we know to be compact and convex. The statement "Hence, the domain is compact and convex" is very, very far away from being obviously true. That the product of any non—empty class of compact spaces is compact, also known as Tychonoff's theorem, to use the words of a famous mathematician, 'is perhaps the most important single theorem of general topology'. (See the brief discussion in the Appendix, Theorem III).

Theorem A. The cross product of two convex sets is convex.

Pf:

Let $(x_1^1, x_2^1)$ and $(x_1^2, x_2^2)$ be two points in $X_1 \times X_2$.

We wish to show that $a (x_1^1, x_2^1) + (1-a) (x_1^2, x_2^2) \in X_1 \times X_2$, $a \in (0,1)$

Now, $a(x_1^1, x_2^1) + (1-a) (x_1^2, x_2^2) = (ax_1^1 + (1-a) x_1^2, ax_2^1 + (1-a) x_2^2)$.

But $x_1^1$ and $x_1^2$ are $\in X_1$, and since $X_1$, is convex we know that $ax_1^1 + (1-a)x_1^2 \in X_1$.

Likewise $x_2^1$, $x_2^2$ are $\in X_2$ and since $X_2$ is convex we know that $ax_2^1 + (1-a) x_2^2 \in X_2$. This implies:

$(ax_1^1 + (1-a) x_1^2, ax_2^1 + (1-a)x_2^2) \in X_1 \times X_2$ and so $X_1 \times X_2$ is convex.

It then follows by induction that the product of any finite class of convex sets is convex.

Tychonoff's theorem together with above theorem then justify the statement that the domain on which the correspondence is defined is compact and convex.

Now, let us make a few observations about our correspondence: P(u(v)) X V(p, w) X W(u(v)).

(1) Theorem 4.6 states that P(u) is compact and convex for fixed u and upper semi continuous in u. Since u(v) is continuous in v (see Section 5.2) it follows that P(u(v)) is compact and convex for fixed v and upper semi—continuous in v.

(2) Theorem 4.5 and Corollary 5 in the same chapter imply that $\hat{W}$ (u(v)) is compact and convex for fixed v and upper semi continous in v.

(3) V(p, w) = $S_H \cap S_B$; where $S_H$ is compact and convex and $S_B$ is closed and convex. Arrow and Hahn state that these properties of $S_H$ and $S_B$ guarantee the compactness of V(p, w) (by Theorem A we know that V(p, w) is convex). To be fully honest and in our desire not to accept anything wthout proof we could attempt to show that either: (a) the intersection of a compact and a closed set is compact or (b) that $S_B$ is also bounded.

None of the two alternatives appeal to me. If we could prove instead that a closed subspace of a compact space is compact then since V(p, w) $\subset S_H$ and V(p, w) is closed (intersection of two closed sets is closed) and $S_H$ is compact it would follow instantly that V(p, w) is compact. This proof is provided in the Appendix (See Theorem I).

The upper semi—continuity of V(p, w) is provided in the text and requires no further elucidation. We only state that the continuity of $s_h$ (p, w) is the crucial tool.

So our correspondence is the cross product of three sets which are compact, coovex and upper semi—continuous. Therefore the correspondance itself is compact, convex and upper semi—continuous. (I must admit that here the reader might say: "you have not proved that the cross product of two upper semi—continuous sets is upper semi—continuoos". True. Lacking a rigorous proof one might appeal to the readers mathematical intuition by saying that taking cross prodncts is itself a continuons operation. One should then not be surprised that such a continuons operation leaves undisturbed the basic properties of the original sets; in this case their upper semi—continuity.

We now are ready to state a theorem which will be of fundamental importance in what remains of this note:

**Kakutani's Fixed Point Theorem.** Let C a compact convex set and g(x) and upper semi—continuous correspondence defined on C such that g(x) $\subset$ C, g(x) convex, for each x in C. Then there exists x* in C such that x* is in g(x*).

Thus there exists a point (p*, v*, w*) $\in s_h$ X $S_H$ X $\hat{W}$ such that

(p*, v*, w*) $\in$ P(u(v*)) X V(p*, w*) X W(u(v*)) (1)

Now write u* = u(v*) and rewrite (1) as:

p* $\in$ P(u*), v* $\in$ V(p*, w*), w* $\in \hat{W}$(u*).

We now proceed to show that (p*, v*, w*) satisfies the conditions for a compensated equilibrium stated at the beginning of this note.

The Conclusions

First, we know that since $u^*$ is Pareto efficient the, by theorem 4.4 there exists a vector p with the following properties: (a) $p > 0$; (b) $p z \geqslant 0$ for all $z \in Z(u^*)$; (c) $p z = 0$ for all $z \in \hat{Z}(u^*)$. Since $p^* \in P(u^*)$ the above theorem implies that $p^* > 0$.

Furthermore since $w^* \in \hat{W}(u^*)$ and $\hat{W}$ is the set of feasible allocations it follows that $w^*$ is feasible (i.e. $z(w^*) \leqslant 0$) which implies:

$$\sum_h x_h^* \leqslant \sum_f y_f^* + \sum_h \bar{x}_h$$

Also, $w^*$ feasible and $p^* \in P(u^*)$ suggest that part (d) of theorem 4.4 is satisfied which, automatically assures that the third and fourth parts of definition I are satisfied. That is:

(c) $y_f^*$ maximizes $p^* y_f$ subject to $y_f \in Y_f$

(d) $x_h^*$ minimizes $p^* x_h$ subject to $U_h(x_h) \geqslant u_h^*$.

Likewise the social budget constraint also holds:

$$\sum_h p^* x_h^* = \sum_h [p^* \bar{x}_h + \sum_f d_{hf} (p^* y_f^*)] \implies$$

$$\sum_h [p^* \bar{x}_h + \sum_f d_{hf} (p^* y^*) - p^* x_h^*] = \sum_h s_h (p^*, w^*) = 0$$

Now, $p^* x_h^* = M_h^* \implies M_h^* - p^* x_h^* = 0$ for all $h \implies s_h(p^*, w^*) = 0$ for all h

We know that it is impossible for all households to incur in budgetary deficits simultaneously since that would imply that $v_h = 0$ for all $h \implies \sum_h V_h = 0 \neq 1$.

Hence the only thing to prove is that $s_h (p^*, w^*) \geqslant 0$ for all h. Suppose not. Then there exists at least one h for which it is true that:

$$s_h (p^*, w^*) < 0 \implies v_h^* = 0 \implies u_h^* = 0.$$

Recall; however, that we have assumed that there exists a possible consumption vector $\bar{\bar{x}}_h$ in $X_h(u_h)$ such that $\bar{\bar{x}}_{hi} \leqslant \bar{x}_{hi}$ for all i, $\bar{\bar{x}}_{hi} < \bar{x}_{hi}$ if $\bar{x}_{hi} > 0$, where $\bar{x}_h$ is the initial endowment for household h. That is, it is possible, (for h), to consume less of good i than household h is endowed with.

So $\bar{\bar{x}}_h \in X_h(u_h^*) = X_h(0)$ and since $x_h^*$ minimizes expenditures $p^* x_h$ subject to $x_h \in X_h(u_h^*) = X_h(0)$ it follows that:

$$p^* x_h^* < p^* \bar{\bar{x}}_h^* < p^* x_h < M_h^* \implies M_h^* - p^* \bar{x}_h > 0 \implies$$

$$s_h (p^*, w^*) > 0 \implies \text{contradiction.}$$

This then establishes the validity of the last condition in our definition of compensated equilibriun, namely that:

$$p^* x_h^* = M_h^*.$$

We can then conclude:

**Theorem B.** Under the assumptions made, a compensated equilibrium exists.

# APPENDIX

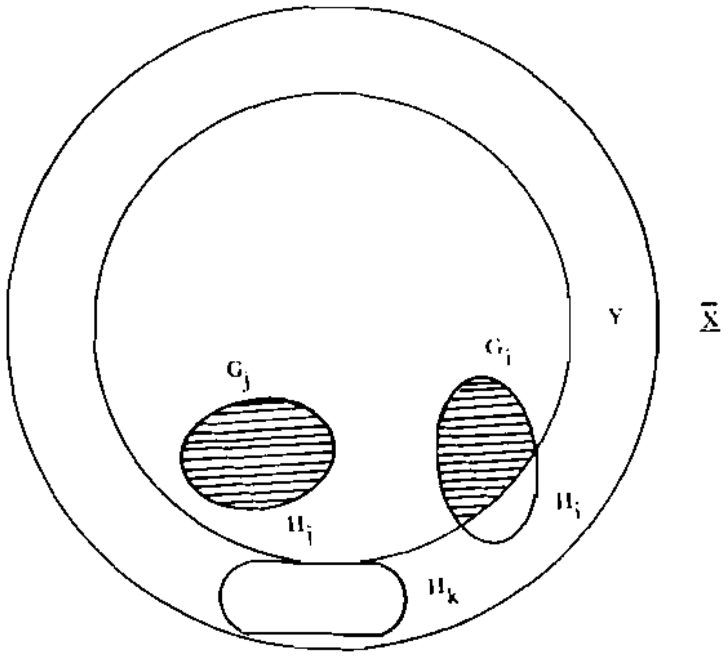
**Definition (a).** Let X be a topological space. A class $\{G_i\}$ of open subsets of X is said to be an "open cover" of X if each point in X belongs to at least one $G_i$; that is $\bigcup_i G_i = X$.

**Definition (b).** A "compact" space is a topological space in which every open cover has a finite subcover.

**Theorem I.** Any closed subspace of a compact space is compact.

**Proof.** Let Y be a close subspace of a compact space X and let the collection of sets $\{G_i\}$ be an open cover of Y. Note that each of the $G_i$'s is open in the relative topology of Y and is the intersection with Y of an open subset $H_i$ of X (See Figure 1). That is $G_i = Y \cap H_i$. Since Y is closed, Y' is open and the class made up of Y' and $\{H_i\}$ is an open cover of X (i.e. given any point in X, say x, then either x is in Y or x is in $H_i$ for some i). But because X is compact this open cover has a finite subcover. If we can show that Y occurs in this subcover we are done. If Y' ocurrs in this subcover we discard it. What is left is a finite class of $H_i$'s whose union contains X. This implies that the correspouding $G_i$'s form a finite subcover of the original open cover of Y.

Figure 1

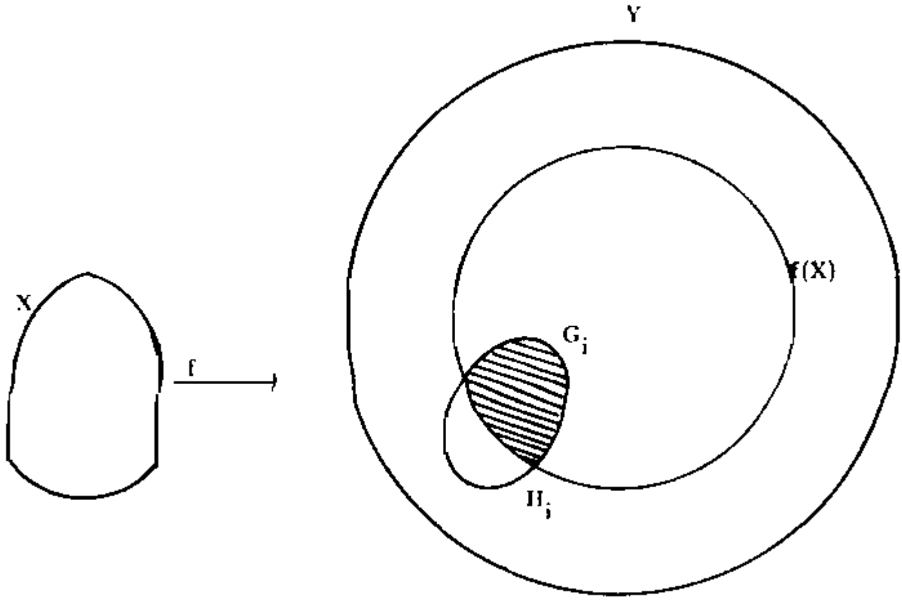



**Theorem II.** Any continuous image of a compact space is compact.

**Proof:** Let $f: X \longrightarrow Y$ be a continuous mapping of a compact space X into an arbitrary topological space Y. We must show that f(X) is a compact subspace of Y.

Let $\{G_i\}$ be an open cover of f(X). As above, each $G_i$ is such that $G_i = f(X) \cap H_i$: that is, it is the intersection with f(X) of an open subset $H_i$ of Y. Recall now that since f is continuous, the inverse image of $H_i$ will be an open set in X; furthermore the set $\{f^{-1}(H_i)\}$ will be an open cover of X and since X is compact it will have a finite subcover. The union of the finite class of $H_i$'s of which the sets in the finite subcover are the inverse images clearly contains f(X), so the class of corresponding $G_i$'s is a finite subcover of the original open cover of f(X). Hence f(X) is compact. (See Figure II).

Figure II




A proof of Tychonoff's Theorem ("The product of any non—empty class of compact spaces is compact") shall not be presented inasmuch as it requires two other difficult theorems: (1) A topological space is compact if and only if every class of closed sets with the finite intersection property has non—empty intersection; and (2) A topological space is compact if very class of subbasic closed sets with the *finite intersection property* has *nonempty intersection.*

A class of subsets of a non—empty set is said to have the "finite intersection property" if every finite subclass has non—empty intersection.

For a proof of Tychonoff's theorem the interest reader may consult, for instance, G.F. Simmons' *"Introduction to Topology and Modern Analysis"* (Mc Graw -Hill), 1963.