Walrasian equilibrium as limit of a competitive equilibrium without divisible goods *

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Abstract

We study economies where all commodities are indivisible at the individual level, but perfectly divisible at the aggregate level of the economy. Under the survival assumption, we show that any rationing equilibrium (Florig and Rivera 2005a) in the discrete economy converges to a Walras equilibrium of the limit economy arising when the level of indivisibility becomes small. If the survival assumption is not satisfied, then rationing equilibrium converges to a hierarchic equilibrium (Florig 2001).

Keywords: competitive equilibrium, indivisible goods, convergence.

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1 Introduction

Perfect divisibility of goods, usually assumed in general equilibrium models, should obviously be seen as an approximation of commodities with a "small" enough level of indivisibility. Florig and Rivera (2005a) define an economy where all goods are indivisible[1] at the individual level but perfectly divisible at the entire economy. Using a parameter called "fiat money" - whose only role is to facilitate the exchange among individuals - and considering a regularized notion of demand, existence of a competitive equilibrium notion called rationing equilibrium is established. Additionally, in a parallel

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[1]See Bobzin (1998) for a survey on indivisible goods.

paper (Florig and Rivera (2005b)) we show that rationing equilibria satisfy First and Second Welfare Theorem and core equivalence, with a refined notion of core due to Konovalov (2005). It is further shown in Florig and Rivera (2005a) that a rationing equilibrium is a Walras equilibrium if the fiat money distribution is in a generic position.

Here we analyze the behavior of the rationing equilibrium when the level of indivisibility becomes small. The study of this asymptotic behavior is motivated by the fact that (i) it will allow us to justify the Walras equilibrium with a convex consumption set as the approximation of a competitive (rationing) equilibrium when the level of indivisibility is small enough and (ii) moreover it shows similarly that hierarchic equilibria (Florig 2001, 2003a) or non standard equilibria introduced by Markulin (1990) using non standard analysis, which exist without a survival assumption, but assuming convex consumption sets, can be viewed as an approximation of a competitive (rationing) equilibrium when the level of indivisibility is small enough. The second point has been studied in the case of linear preferences in Florig (2003b).

The nature of the equilibrium at the limit economy will depend strongly on the assumptions made on it. We will prove that if both the survival assumption and local non-satiation hold true at the limit economy, which of course are common assumptions in the standard Arrow-Debreu model (see Arrow and Debreu (1954)), then the rationing equilibrium converges to Walras equilibrium. If the survival assumption does not hold, then the rationing equilibrium converges to a hierarchic equilibrium. We recall that at a hierarchic equilibrium, consumers are partitioned according to their level of wealth: poorer consumers have not access to all the expensive commodities to which the richer have access. Such access restrictions occur easily if the commodities are not perfectly divisible, as we described above. So the same phenomena occur as in the case of indivisible economies, and for the same reason only a weak version of Pareto optimality holds. This formally confirms the interpretation of hierarchic equilibria in terms of small indivisibilities given in Florig (2001, 2003b).

This work is organized as follows. In Section 2 we briefly describe the model developed in Florig and Rivera (2005a). In Section 3 we introduce the convergence concept for economies and present our first main result (Propositions 3.1), which give us conditions to assure that the limit equilibrium is a Walras one. In Section 4 we consider a more general framework than in Section 3. The main result of this section is Theorem 2.1, which establishes that for a limit economy with neither the survival assumption nor a local non-satiation hypothesis, the asymptotic equilibrium is a hierarchic one. Proposition 3.1 is obtained as a corollary of this Theorem.

2 Model

For details, interpretation and proofs on the model we will present in this section, we refer to Florig and Rivera (2005a). Thus, we set $L \equiv \{1, \ldots, L\}$, $I \equiv \{1, \ldots, I\}$ and $J \equiv \{1, \ldots, J\}$ to denote the finite set of commodities, the finite sets of types of consumers and producers, respectively. We assume that each type $k \in I, J$ of agents consists of a continuum of identical individuals indexed by a set $T_k \subset I\!\!R$ of finite

Lebesgue measure[2]. We set $\mathcal{I} = \cup_{i\in I} T_i$ and $\mathcal{J} = \cup_{j\in J} T_j$. Of course, $T_k \cap T_{k'} = \emptyset$ if $k \neq k'$. Given $t \in \mathcal{I}$ ($\mathcal{J}$), let

$$i(t) \in I \ (j(t) \in J)$$

be the index such that $t \in T_{i(t)}$ ($t \in T_{j(t)}$).

Each firm of type $j \in J$ is characterized by a finite production set[3] $Y_j \subset I\!\!R^L$ and the aggregate production set of firms of type $j \in J$ is the convex hull of $\mathcal{L}(T_j)Y_j$, which is denoted by $\mathrm{co}\,[\mathcal{L}(T_j)Y_j]$.

Every consumer of type $i \in I$ is characterized by a finite consumption set $X_i \subset R^L$, an initial endowment $e_i \in I\!\!R^L$ and a strict preference correspondence $P_i : X_i \to X_i$. Let $e = \sum_{i\in I} \mathcal{L}(T_i)e_i$ be the aggregate initial endowment of the economy and for $(i,j) \in I \times J$, $\theta_{ij} \geq 0$ is the share of type $i \in I$ consumers in type $j \in J$ firms. For all $j \in J$, assume that $\sum_{i\in I} \mathcal{L}(T_i)\theta_{ij} = 1$.

The initial endowment of fiat money for an individual $t \in \mathcal{I}$ is defined by $m(t)$, where $m : \mathcal{I} \to I\!\!R_+$ is a Lebesgue-measurable and bounded mapping.

Given all the above, an *economy* $\mathcal{E}$ is a collection

$$\mathcal{E} = \Big((X_i, P_i, e_i, m)_{i\in I}, (Y_j)_{j\in J}, (\theta_{ij})_{(i,j)\in I\times J}\Big),$$

an *allocation* (or consumption plan) is an element of

$$X = \Big\{x \in \mathcal{L}^1(\mathcal{I}, \cup_{i\in I} X_i) \,|\, x_t \in X_{i(t)} \text{ for a.e. } t \in \mathcal{I}\Big\},$$

a *production plan* is an element of

$$Y = \Big\{y \in \mathcal{L}^1(\mathcal{J}, \cup_{j\in J} Y_j) \,|\, y_t \in Y_{j(t)} \text{ for a.e. } t \in \mathcal{J}\Big\},$$

and the *feasible consumption-production plans* are elements of

$$A(\mathcal{E}) = \Big\{(x,y) \in X \times Y \mid \int_{\mathcal{I}} x_t = \int_{\mathcal{J}} y_t + e\Big\}.$$

In the rationing equilibrium definition below we will employ *pointed cones* in $I\!\!R^L$, which is the set of convex cones $\mathcal{C} \subseteq I\!\!R^L$ such that $K \in \mathcal{C}$ if and only if $-K \cap K = \{0_{I\!\!R^L}\}$.

Given $p \in I\!\!R^L_+$, let us define the *supply* and *profit* of a type $j \in J$ firm as

$$S_j(p) = \mathrm{argmax}_{y\in Y_j}\ p \cdot y \qquad\qquad \pi_j(p) = \mathcal{L}(T_j)\mathrm{sup}_{y\in Y_j}\ p \cdot y$$

and given additionally $K \in \mathcal{C}$ we define the rationing supply (in the following simply *supply*) for a firm $t \in \mathcal{J}$ by

$$\sigma_t(p,K) = \{y \in S_{j(t)}(p) \,|\, (Y_{j(t)} - y) \cap p^{\perp} \subset -K\}.$$

[2]Without loss of generality we may assume that T_k is a compact interval of $I\!\!R$. In the following, we note by $\mathcal{L}(T_k)$ the Lebesgue measure of set $T_k \subseteq I\!\!R$. Finally, we denote by $\mathcal{L}^1(A,B)$ the Lebesgue integrable functions from $A \subset I\!\!R$ to $B \subset I\!\!R^L$.

[3]That is, the number of admissible production plans for the firm is finite.

For prices $(p,q) \in I\!R^L \times I\!R_+$, we denote the *budget set* of a consumer $t \in \mathcal{I}$ by $B_t(p,q) = \left\{x \in X_{i(t)} \,|\, p \cdot x \leq w_t(p,q)\right\}$ where $w_t(p,q) = p \cdot e_{i(t)} + qm(t) + \sum_{j \in J} \theta_{i(t)j}\pi_j(p)$ is the wealth of individual $t \in \mathcal{I}$. The set of *maximal elements* for the preference relation in the budget set for consumer $t \in \mathcal{I}$ is denoted by $d_t(p,q)$ and given that, we define the weak demand at the respective prices as

$$D_t(p,q) = \limsup_{(p',q') \to (p,q)} d_t(p',q').$$

Previous concept is used to define our notion of *demand*, which for a cone $K \in \mathcal{C}$ and prices $(p,q) \in I\!R^L \times I\!R_+$ is defined as

$$\delta_t(p,q,K) = \{x \in D_t(p,q) \,|\, (P_t(x) - x) \cap p^{\perp} \subset K\}.$$

Remark 2.1 In Florig and Rivera (2005a) it is proven that if $qm_t > 0$ then

$$D_t(p,q) = \{x \in B_t(p,q) \mid p \cdot P_{i(t)}(x) \geq w_t(p,q),\ x \notin \mathrm{co}P_{i(t)}(x)\}.$$

With the previous concepts, we can now define our equilibrium notions.

Definition 2.1 Let $(x,y,p,q) \in A(\mathcal{E}) \times I\!R^L \times I\!R_+$ and $K \in \mathcal{C}$.

We call (x,y,p,q) a *Walras equilibrium* with money of $\mathcal{E}$ if for a.e. $t \in \mathcal{I}$, $x_t \in d_t(p,q)$ and for a.e. $t \in \mathcal{J}$, $y_t \in S_{j(t)}(p)$.

We call (x,y,p,q,K) a *rationing equilibrium* of $\mathcal{E}$ if for a.e. $t \in \mathcal{I}$, $x_t \in \delta_t(p,q,K)$ and for a.e. $t \in \mathcal{J}$, $y_t \in \sigma_t(p,K)$.

We call (x,y,p,q) a *rationing equilibrium* of $\mathcal{E}$ if for a.e. $t \in \mathcal{I}$, $x_t \in D_t(p,q)$ and for a.e. $t \in \mathcal{J}$, $y_t \in S_{j(t)}(p)$.

Remark 2.2

a.- Note that every Walras equilibrium is a rationing equilibrium and a rationing equilibrium is a weak equilibrium.

b.- In general, it is well known that Walras equilibrium fails to exists when goods are indivisible. Mathematically this cames from the fact that the correspondence d_i is not necessarily upper semi continuous with respect to (p,q), which oblige us to define a regularized notion of it, D_i.

c.- Weak equilibrium is an auxiliary equilibrium concept which is crucial building block for the existence proof of the rationing equilibrium. We will establish our main results for weak equilibria, which off course implies that they hold true for rationing equilibria (and Walras equilibria when they exist in the discrete economy). For more details see Florig and Rivera (2005a).

The following proposition is proven in Florig and Rivera (2005a).

Theorem 2.1 If for all $i \in I$, P_i is irreflexive and transitive and under

Assumption S. For all $i \in I$,

$$0 \in \text{int}(X_i - \{e_i\} - \sum_{j \in J} \theta_{ij} \mathcal{L}(T_j) Y_j).$$

then there exists a weak equilibrium with $q > 0$. Moreover if $m_i > 0$ for all $i \in I$, there exists a rationing equilibrium with $q > 0$.

3 Convergence: the simple case

In this section we consider what we call the simple case, which is when the limit economy fulfill the standard survival assumptions.

We begin defining formally what we understand as convergence of economies (which is also valid for the general case in next Section). To do so, we employ the Kuratowski-Painlevé notion of set convergence (we refers to Rockafellar and Wets (1998) for details on this concept).

Given a sequence of sets $\{Z_k\} \subset I\!R^m$, we recall that the upper limit of this sequence is defined as the set

$$\limsup_{k \to \infty} Z_k := \{x \in I\!R^\ell \mid \exists\ x_{k'} \to x, x_{k'} \in Z_{k'}\}$$

whereas the its lower limit is defined as

$$\liminf_{k \to \infty} Z_k := \{x \in I\!R^\ell \mid \forall\ k \in I\!N\ \exists\ x_k \in Z_k,\ x_k \to x\}.$$

Given that, we say that the family of sets $\{Z_k\} \subset I\!R^m$ converges in the sense of Kuratowski-Painlevé to a set $Z \subset I\!R^m$ if

$$\limsup_{k \to \infty} Z_k = \liminf_{k \to \infty} Z_k = Z.$$

Definition 3.1

a.- We say that a sequence $n = (n_1, \cdots, n_L) \subset I\!N^L$ *converges to* $+\infty$, if for all $h \in L$, n_h converges to $+\infty$.

b.- For every $n = (n_1, \ldots n_L) \in I\!N^L$, let

$$M^n = \{z \in I\!R^L \,|\, (n_1 z_1, \ldots, n_L z_L) \in Z^L\}.$$

We recall that, according to previous definitions, it is easy to check that M^n converges in the sense of Kuratowski-Painlevé to $I\!R^L$.

Thus, given the economy

$$\mathcal{E} = ((X_i, P_i, \omega_i, m_i)_{i\in I}, (Y_j)_{j\in J}, (\theta_{ij})_{(ij)\in I\times J}),$$

for $n \in I\!N^L$ and M^n as above, let us consider the following definition of a sequence of economies that approximate $\mathcal{E}$.

Definition 3.2 A sequence of discreet economies that approximate $\mathcal{E}$ is defined as

$$\mathcal{E}^n = ((X_i^n, P_i, \omega_i, m_i)_{i\in I}, (Y_j^n)_{j\in J}, (\theta_{ij})_{(ij)\in I\times J})$$

where for every $i \in I$ and $j \in J$

$$X_i^n = X_i \cap M^n, \quad Y_j^n = Y_j \cap M^n$$

and for every $n \in I\!N^L$, M^n contains all of the extremal points of polyhedra[4] X_i and Y_j for each $i \in I$ and $j \in J$.

We denote the weak supply, budget and weak demand in the economy $\mathcal{E}^n$ by S_j^n, B_i^n, D_i^n respectively.

To prove the main result of this section, we need following assumptions.

Assumption C. For all $i \in I$, X_i is a compact, convex polyhedron, $P_i : X_i \to 2^{X_i}$ is irreflexive, transitive and has an open graph in $X_i \times X_i$.

Assumption P. For every $j \in J$, Y_j is a compact, convex polyhedron.

Assumption SS. For all $i \in I$,

$$0 \in \mathrm{int}(X_i - \{e_i\} - \sum_{j\in J} \theta_{ij}\mathcal{L}(T_j)Y_j).$$

Assumption M. For a.e. $t \in T_i$,

$$m(t) = m_i > 0.$$

Proposition 3.1 *Suppose $\mathcal{E}$ satisfies Assumptions C, P, SS, M and let $\mathcal{E}^n$ be a sequence of economies that approximate $\mathcal{E}$. Let (x^n, y^n, p^n, q^n) be a weak equilibrium of $\mathcal{E}^n$ with $q^n > 0$ and such that $(p^n, q^n) \to (p, q)$, for a.e. $t \in \mathcal{I}$, $x_t \in \mathrm{cl}x_t^n$ and for a.e. $t \in \mathcal{J}$, $y_t \in \mathrm{cl}y_t^n$. Then (x, y, p, q) is a Walras equilibrium with fiat money for $\mathcal{E}$ and if for a.e. $t \in \mathcal{I}$, $x_t \in \mathrm{cl}P_{i(t)}(x_t)$ (local non-satiation holds at x_t) then (x, y, p) is a Walras equilibrium for $\mathcal{E}$.*

Proof. This proposition comes directly from Theorem 2.1 considering that if Assumption SS holds then a hierarchic equilibrium is a Walras equilibrium with fiat money and a Walras equilibrium if local non satiation is satisfied (cf Florig (2001)). □

[4] We have not checked weather this condition is actually needed, but it simplifies the proof.

Remark 3.1 In previous proposition, survival assumption SS plays an important role in establishing the convergence to a Walras equilibrium. However, although this hypothesis is widely used in economic theory to prove existence, it is utterly unrealistic because it states that every consumer is initially endowed with a strictly positive quantity of every existing commodity. Typically, most consumers have a single commodity to sell - their labor. In fact, it implies that all agents have the same level of income at equilibrium in the sense that they have all access to the same commodities. In next section we replace this hypothesis by a more realistic assumptions, i.e. we will assume that every consumer could decide not to exchange anything. We will not assume however that he could survive for very long without exchanging anything. In such a case the limit allocation will not necessarily be a Walras equilibrium. It will be a hierarchic equilibrium, which is a competitive equilibrium with a segmentation of individuals according to their level of wealth. In the particular case that this segmentation consists in just one group, the hierarchic equilibrium reduces to a Walras equilibrium.

4 The general case

Following Florig (2001), we will now introduce the notion of hierarchic equilibrium[5]. Let $\bar{I\!R} = (I\!R \cup \{+\infty\})$. For any $n \in N$, let $\preceq$ be the lexicographic order[6] on $I\!R^n$. Extrema will be taken with respect to the lexicographic order. We adopt the convention $0(+\infty) = 0$.

Definition 4.1 A finite ordered family $\mathcal{P} = \{p^1, \dots, p^k\}$ of vectors of $I\!R^L$ is called a hierarchic price.

Remark 4.1 If $k = 1$, this reduces to the standard case. We denote by $\mathcal{HP}$ the set of hierarchic prices. The number k is determined at the equilibrium. We will see that k never needs to be greater than L.[7]

For $\mathcal{P} \in \mathcal{HP}$ and $x \in I\!R^L$, we define the *value* of x to be

$$\mathcal{P}x = (p^1 \cdot x, \dots, p^k \cdot x) \in I\!R^k.$$

The *supply* of firm $j \in J$ at the price $\mathcal{P}$ is

$$S_j(\mathcal{P}) = \{y \in Y_j \mid \forall z \in Y_j,\ \mathcal{P}z \preceq \mathcal{P}y\}.$$

Given a hierarchic price, firms are thus assumed to maximize the profit lexicographically. The aggregate *profit* of firms of type $j \in J$ is

$$\pi_j(\mathcal{P}) = \mathcal{L}(T_j)\mathrm{sup}_{y \in Y_j}\mathcal{P}y.$$

[5]Marakulin (1990) introduced a similar notion for exchange economies, using non-standard analysis.

[6]For $(s,t) \in I\!R^n \times I\!R^n$, $s \preceq t$, if $s_r > t_r$, $r \in \{1, \dots, n\}$ implies that $\exists \rho \in \{1, \dots, r-1\}$ such that $s_\rho < t_\rho$. We write $s \prec t$ if $s \preceq t$, but not $[t \preceq s]$.

[7] The forthcoming definitions will depend for any $r \in \{2, \dots, k\}$ only on the non-zero part of p^r which is orthogonal to $p^1, \dots, p^{r-1}$. Therefore by an inductive argument we can always transform a hierarchic price into an equivalent one consisting of two by two orthogonal vectors (thus of at most L).

A *hierarchic revenue* is a vector $w \in I\!R^k$. For all $t \in \mathcal{I}$, all $\mathcal{P} \in \mathcal{HP}$, all $w \in I\!R^k$ let

$$r_t(\mathcal{P}, w) = \min\{r \in \{1, \ldots, k\} \mid \exists x \in X_i(t), (p^1 \cdot x, \ldots, p^r \cdot x) \prec (w^1, \ldots, w^r)\},$$

$$v_t(\mathcal{P}, w) = (w^1, \ldots, w^{r_t(\mathcal{P},w)}, +\infty, \ldots, +\infty) \in \bar{I\!R}^k.$$

The *budget set* of consumer t, with respect to $\mathcal{P} \in \mathcal{HP}$ and $w \in I\!R^k$ will be

$$B_t(\mathcal{P}, w) = \{x \in X_{i(t)} \mid \mathcal{P}x \preceq v_t(\mathcal{P}, w)\}.$$

Definition 4.2 [8] A collection $(x, y, \mathcal{P}, \mathcal{Q}) \in A(\mathcal{E}) \times \mathcal{HP} \times \cup_{k=1}^{L} I\!R^k_+$ is a **hierarchic equilibrium** of the economy $\mathcal{E}$ if:

(i) for a.e. $t \in \mathcal{I}$, $x_t \in B_t(\mathcal{P}, w_t)$ and $P_t(x_t) \cap B_t(\mathcal{P}, w_t) = \emptyset$;

(ii) for all $t \in \mathcal{I}$, $\mathcal{P}e_{i(t)} + \sum_{j \in J} \theta_{i(t)j} \pi_j(\mathcal{P}) + Qm_t = w_t$;

(iii) for a.e. $t \in \mathcal{J}$, $y_t \in S_t(\mathcal{P})$.

Now we are conditions to prove the main theorem of this paper, which as we mentioned, is a generalization of Proposition 3.1.

Theorem 4.1 Suppose $\mathcal{E}$ satisfies Assumptions C, P, S, M. Consider a sequence $n \subset I\!N^L$ converging to ∞ such that for all n, for all $i \in I$, $X_i = \mathrm{co}(X_i \cap M^n)$ and for all $j \in J$, $Y_j = \mathrm{co}(Y_j \cap M^n)$.

Let (x^n, y^n, p^n, q^n) be a weak equilibrium of $\mathcal{E}^n$ with $q^n > 0$. Then, there exists a hierarchic equilibrium $(x, y, \mathcal{P}, \mathcal{Q})$ with $\mathcal{P} = \{p^1, \ldots, p^k\}$, $\mathcal{Q} = (q^1, \ldots, q^k)$ and a subsequence such that:

- For a.e. $t \in \mathcal{I}$ and a.e. $t' \in \mathcal{J}$,[9]

$$x_t \in \mathrm{cl}\{x^n_t\} \text{ and } y_{t'} \in \mathrm{cl}\{y^n_{t'}\};$$

- $p^n = \sum_{r=1}^{k} \varepsilon^n_r p^r$, with $\varepsilon^n_{r+1} = \varepsilon^n_r o(\varepsilon^n_r) > 0$[10] and $\lim_{n \to +\infty} \varepsilon^n_1 = 1$;
- $w_i = \mathcal{P}e_{i(t)} + \sum_{j \in J} \theta_{ij} \pi_j(\mathcal{P}) + \{q_1, \ldots, q_k\} m_t$ with $q^1 = \ldots, q^{k-1} = 0$ and $q^k \geq 0$.

Proof Let (x^n, y^n, p^n, q^n) be a weak equilibrium of $\mathcal{E}^n$ with $q^n > 0$. Without loss of generality assume that $q^n = (1 - \parallel p^n \parallel) > 0$. For all $i \in I$, $j \in J$ note $\bar{x}^n = \int_{T_i} x^n_i / \mathcal{L}(T_i)$ and $\bar{y}^n = \int_{T_j} y^n_j / \mathcal{L}(T_j)$ the average consumption and production plans per type at x^n, y^n respectively. We note

$$\beta_i(p) = \{x \in X_i \,|\, p \cdot x \leq p \cdot \omega_i + (1 - \|p\|)m_i + \sum_{j \in j} \theta_{ij} \pi_j(p)\}.$$

As in Florig (2001, 2003a), we can extract a subsequence such that

[8] If we note L_i the linear space of the positive hull generated by consumer i's net trade set and c_i the codimension of L_i, then we may reduce any hierarchic price into an equivalent one with $k \leq 1 + \min_{i \in I} c_i$. Indeed, either 0 is an equilibrium price or we may assume the prices two by two orthogonal and all non-zero (cf. Footnote 7). The rank of consumer i is smaller than or equal to the index of the first vector which is not orthogonal to L_i. The prices of a higher index are irrelevant to this consumer.

[9] We note cl Z for the closure of Z.

[10] Let $o : I\!R_+ \to I\!R_+$ such that $o(0) = 0$ and o is continuous in 0.

- $p^n = \sum_{r=1}^{k} \varepsilon_r^n p^r$, with $\varepsilon_{r+1}^n = \varepsilon_r^n o(\varepsilon_r^n) > 0$ for $r \in \{1, \ldots, k-1\}$, and $\lim_{n\to\infty} \varepsilon_1^n = 1$. Let $\mathcal{P} = \{p^1, \ldots, p^k\}$;

- for all $r < k$, $\frac{q^n}{\varepsilon_r^n}$ converges to 0 and $\frac{q^n}{\varepsilon_k^n}$ converges in $I\!R_+$;

- for all large enough n, for all $j \in J$, $\mathrm{co}S_j^n(p^n) = S_j(\mathcal{P})$ and thus $\bar{y}_j^n \in S_j(\mathcal{P})$ and $\bar{y}_j \in S_j(\mathcal{P})$;

- for a.e. $t \in \mathcal{I}$, $\beta_t(p^n)$ converges to $B_t(\mathcal{P}, w_t)$.

For the last two points, we use the fact that for all n, for all $j \in J$, $\mathrm{co}Y_j^n = Y_j$ and for all $i \in I$, $\mathrm{co}X_i^n = X_i$, in order to apply the arguments from Florig (2001, 2003a).

Since the consumption sets are compact and for all n, $(x^n, y^n) \in A(\mathcal{E}^n)$, there exists by Fatou's lemma (Arstein (1979)) $(x, y) \in A(\mathcal{E})$ such that for a.e. $t \in \mathcal{I}$ and a.e. $t' \in \mathcal{J}$,

$$x_t \in \mathrm{cl}\{x_t^n\} \text{ and } y_{t'} \in \mathrm{cl}\{y_{t'}^n\}.$$

Thus, by the second point above for a.e. $t \in \mathcal{J}$, $y_t \in S_{j(t)}(\mathcal{P})$. Obviously, for a.e. $t \in \mathcal{I}$, $x_t \in \lim_{n\to\infty} B_{i(t)}^n(p^n, q^n)$. Moreover, $\lim_{n\to\infty} B_{i(t)}^n(p^n, q^n) \subset B_{i(t)}(\mathcal{P}, w_t)$.

It remains to be proven that for a.e. $t \in \mathcal{I}$, $P_{i(t)}(x_t) \cap B_{i(t)}(\mathcal{P}, w_t) = \emptyset$. We will proceed by contraposition. Let $\mathcal{N}$ be the negligible subset of $\mathcal{I}$ containing all $t \in \mathcal{I}$ such that either for some n, $x_t^n \notin D_{i(t)}^n(p^n, q^n)$ or such that $x_t \notin \mathrm{cl}\{x_t^n\}$. This set is negligible since it is a countable union of negligible sets. Let $t \in \mathcal{I} \setminus \mathcal{N}$ such that there exists $\xi_t \in P_{i(t)}(x_t) \cap B_{i(t)}(\mathcal{P}, w_t)$. If the budget set is reduced to a single point then $x_t = \xi_t$. Thus the budget set has a non-empty interior in some facet F of $X_{i(t)}$. By the continuity of $P_{i(t)}$, we may assume that $\xi_t \in \mathrm{int}_F(F \cap B_{i(t)}(\mathcal{P}, w_t))$ and since $F \cap M^n$ converges to F, we may assume for all large n, $\xi_t \in M^n \cap X_{i(t)}$. If $F \subset B_{i(t)}(\mathcal{P}, w_t)$, then since for all n, $\mathrm{int}_{X_{i(t)}} \beta_t(p^n) \neq \emptyset$, we have for all large n,

$$p^n \cdot \xi_t < w_t^n = p^n \cdot e_{i(t)} + (1 - \|p^n\|) m_{i(t)} + \sum_{j \in j} \theta_{i(t)j} \pi_j(p^n).$$

If $F \not\subset B_{i(t)}(\mathcal{P}, w_t)$, there exists $\xi_t' \in \mathrm{int}_F(F \cap B_{i(t)}(\mathcal{P}, w_t))$ such that $\mathcal{P}\xi_t \prec \mathcal{P}\xi_t'$. Therefore for all large n, $p^n \cdot \xi_t < p^n \cdot \xi_t'$ and then we have again for all large enough n, $p^n \cdot \xi_t < w_t^n$. By the continuity of $P_{i(t)}$, if $p^n \cdot \xi_t < w_t^n$ for a subsequence, we have $\xi_t \in P_{i(t)}(x_t^n)$ for this subsequence. Thus $x_t^n \notin D_{i(t)}^n(p^n, q^n)$, a contradiction. □

References

[1] Arrow, K.J. and G. Debreu (1954): "Existence of an Equilibrium for a Competitive Market," *Econometrica*, 22, 265-290.

[2] Arstein, A. (1974): "A Note on Fatou's Lemma in Several Dimensions," *Journal of Mathematical Economics*, 6 , 277-282.

[3] Bobzin, H. (1998): *Indivisibilities: Microecomic Theory with Respect to Indivisible Goods and Factors*, Physica Verlag, Heidelberg.

[4] Drèze, J.H. and H. Müller (1980): "Optimality Properties of Rationing Schemes," *Journal of Economic Theory*, 23, 150-159.

[5] Florig, M. (2001): "Hierarchic Competitive Equilibria," *Journal of Mathematical Economics*, 35(4), 515-546.

[6] Florig, M. (2003a): "Corrections to Hierarchic Competitive Equilibria," Cahiers de la MSE, Série Bleue 2003.65, Université de Paris 1, available on http://www.florig.de.

[7] Florig, M. (2003b): "Arbitrary Small Indivisibilities", *Economic Theory*, 22, 831-843.

[8] Florig, M. and J. Rivera (2005a):" Indivisible goods and fiat money". Working Paper $N°$ 212, Departamento de Economía, Universidad de Chile.

[9] Florig, M. and J. Rivera (2005b):" Welfare properties and core for a competitive equilibrium without divisible goods". Working Paper 213, Departamento de Economía, Universidad de Chile.

[10] Kajii, A. (1996): "How to Discard Non-Satiation and Free-Disposal with Paper Money," *Journal of Mathematical Economics*, 25, 75-84.

[11] Konovalov, A. (2005): "The core of an economy with satiation", *Economic Theory*, 25, 711-719.

[12] Marakulin, V. (1990): "Equilibrium with Nonstandard Prices in Exchange Economies," in *Optimal decisions in market and planned economies*, edited by R. Quandt and D. Triska, Westview Press, London, 268-282.

[13] Rockafellar, R. and Wets, R. (1998): *Variational Analysis*. Springer.