Welfare Theorema and Core Equivalence without Divisible Goods*

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We study economies where all commodities are indivisible at the individual level, but perfectly divisible at the aggregate level. Paper (fiat) money which does not influence agents preferences may be used to facilitate exchange. In a parallel paper (Florig and Rivera (2002), we introduced a competitive equilibrium notion for such a set up called rationing equilibrium. Here, we will establish welfare theorema and a core equivalence result for this equilibrium notion.

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1.- Introduction

In general equilibrium theory, it is well known that a Walras equilibrium may fail to exist in the presence of indivisible goods and even the core may be empty as it is shown in Shapley and Scarf (1974). Numerous authors (Brome (1972), Mas Colell (1977), Kahn and Yamazaki (1981), Quinzii (1984), see Bobzin (1998) for a survey) consider economies with indivisible commodities and one perfectly divisible. All these contributions suppose that the divisible commodity satisfies overriding desirability, i.e. it is so desirable by the agents that it can replace the consumption of indivisible goods. Moreover, every agent initially owns an important quantity of this good. In such case, the non-emptiness of the core and existence of a Walras equilibrium is then ensured.

In the model developed in Florig and Rivera (2002) it is assumed that all the consumption goods are indivisible at individual level but perfectly divisible at the aggregate level. In such case it is not possible to show nonemptiness of core and the existence of a Walras equilibrium in a general case. However, it can be proved the existence of the so called *rationing equilibrium*, which is a competitive notion. In fact, depending on the type of survival assumption made on, convergence to either a Walras or hierarchc (Florig (2001)) equilibrium holds when indivisibilities *became small*. However, in spite of these results, it is still open the question on Welfare and core equivalence of rationing equilibrium, whose study is the objective of this paper.

As one of the main results of this work, we are going to demonstrate that rationing equilibrium is weakly Pareto optimal and is in the Konovalov's (1998) rejective core (which is a refinement of the standard weak core). In fact, strong Pareto optimally fails due to the fact that some consumers may own commodities which are worthless to them as a consumption good (or they own more than they need). Under indivisible goods in the economy, the value of these commodities may be so small that selling them does not enable to by more of the goods they are interested. Thus, they may waste these commodities, which could however be very useful and expensive for poorer agents. So the market is not as efficient as in the standard Arrow-Debreu setting (Arrow and Debreu (1954)).

2.- Model and preliminaries

2.1. Basic concepts

For details, interpretation and proofs on the model we are going to present in this section, we refer to Florig and Rivera (2002).

We set L to denote the finite set of commodities and let I, J be finite sets of types of identical consumers and firms respectively. We assume that each type $k \in I, J$ of agents consists of a continuum of identical individuals represented by a compact interval $T_k \subset IR$, whose length is $\lambda(T_k)$. We set $\mathcal{I} = \cup_{i \in I} T_i$ and $\mathcal{J} = \cup_{j \in J} T_j$. Of course $T_t \cap T_{t'} = \phi$ if type t and t' are different.

Each firm of type $j \in J$ is characterized by a **finite** production set $Y_j \subset IR^L$ and the aggregate production set of type j firms in the convex hull $\lambda(T_j)\ coY_j$. Every consumer of type $i \in I$ is characterized by a **finite** consumption set $X_i \subset R^L$, an initial endowment of goods $e_i \in IR^L$, an initial endowment of *paper money* $m_i \geq 0$ [1] and a preference correspondence $P_i : X_i \to 2^{X_i}$. Let $e = \sum_{i \in I} \lambda(T_i)\ e_i$ be the aggregate initial of the economy. For $(i, j) \in I \times J, \theta_{ij} \geq 0$ is the share of type i consumers in type j firms. For all $j \in J, \sum_{i \in I} \lambda(T_i)\theta_{ij} = 1$.

With all foregoing, an *economy* ξ is a collection

$$\varepsilon = \left((X_i, P_i, e_i, m_i)_{i \in I}, (Y_j)_{j \in J}, (\theta_{ij})_{(i,j) \in I \times J}\right).$$

Feasible consumption-production plans are elements of

$$A(\mathcal{E}) = \left\{(x, y) \in X \times Y \mid \textstyle\int_{\mathcal{I}} x_t = \int_{\mathcal{J}} y_t + e\right\}$$

1 In the model is assumed that this "good" is perfectly divisible, may be consumed in positive amounts, but does not enter the consumers preferences.

where[2]

$$X = \{x \in L^1(\mathcal{I}, \cup_{i \in I} X_i) \mid x_t \in X_t \text{ for a.e. } t \in \mathcal{I}\}$$

$$Y = \{y \in L^1(\mathcal{J}, \cup_{j \in J} Y_j) \mid y_t \in Y_t \text{ for a.e. } t \in \mathcal{J}\}.$$

We denote by C the set of pointed cones[3] $K \subset IR^L$ and given $(p,q) \in IR^L \times IR_+$ we have the following definition.

Definition 2.1.

a.- For a type $i \in I$ *consumer, we define*

(i) *Budget set:* $B_i(p,q) = \{\chi \in X_i | p \cdot \chi \le p \cdot e_i + qm_i + \sum_{j \in J} \theta_{ij} \pi_j(p)\}$.

(ii) *Walrasian demand:* $d_i(p,q) = \{\chi_i \in B_i(p,q) | B_i(p,q) \cap P_i(\chi) = \theta\}$

(v) *Weak demand:* $D_i(p,q) = \limsup_{(p',q') \to (p,q)} d_i(p',q')$

(vi) *Rationing demand:* $\delta_i(p,q,K) = \{\chi \in D_i(p,q) | P_i(\chi) - \chi \subset K\}$

b.- For a type $j \in J$ *firm we define*

(i) *Walrasian supply:* $S_j(p) = \arg\max_{y \in Yj} p \cdot y$

(iii) *Aggregate Walrasian profit:* $\pi_j(p) = \lambda(T_j) \sup_{y \in Yj} p \cdot y$

(vii) *Rationing supply:* $\sigma_j(p,K) = \{y \in S_j(p) \; | Y_t - y \subset -K\}$

2 We note $L^1(A,B)$ the Lebesgue integrable functions from A to B

3 A cone K is pointed if it contains no straight line, i.e. $-K \cap K = \{0\}$. See Rockafellar and Wets (1998).

With previous concepts, we are going to define our equilibrium notions. Thus, given $(\chi, y, p, q) \in A\ (\varepsilon) \times IR^L \times IR_+$ and $K \in C$, we have that.

Definition 2.2. $a.-(\chi, y, p, q)$ is a **Walras equilibrium** *with money of* $\mathcal{E}$ *if*

a.- *(i)* *for a.e.* $t \in \mathcal{I},\ x_t \in d_t(p, q);$

(ii) *for a.e.* $t \in \mathcal{J},\ y_t \in S_t(p).$

b.- (χ, y, p, q) *is a* **weak equilibrium** of $\mathcal{E}$ if

(i) *for a.e.* $t \in \mathcal{I},\ x_t \in D_t(p, q);$

(ii) *for a.e.* $t \in \mathcal{J},\ y_t \in S_t(p).$

c.- (χ, y, p, q, K) *is a* **rationing equilibrium** of $\mathcal{E}$ if

(i) *for a.e.* $t \in \mathcal{I},\ x_t \in \delta_t(p, q, K);$

(ii) *for a.e.* $t \in \mathcal{J},\ y_t \in \sigma_t(p, K).$

Remark 2.1.

a.- Note that every Walras equilibrium is a weak equilibrium and every weak equilibrium is a rationing equilibrium.

b.- In general, it is well known that Walras equilibrium fails to exists when goods are indivisibile (Shapley and Scarf (1974)). Mathematically this cames from the fact that demand correspondence d_i is not upper semi continuous with respect to (p, q) and this is the reason why we defined a regularized notion of it (D_i).

c.- Weak equilibrium must be seen as an auxiliary equilibrium concept which is crucial building block for the existence proof, Welfare properties and core equivalence of our full blown equilibrium concept (rationing equilibrium).

d.- An economical interpretation of weak demand (and then rationing demand) is given in Florig and Rivera (2002). There is proved that if $qm_i\rangle 0$ [4] then.

$$D_i(p,q)=\{\chi\in B_i(p,q)\ |p\cdot P_i(\chi)\geq\omega_i(p,q),\chi\notin coP_i(\chi)\}.$$

Finally, following proposition (existence of equilibrium) is proved in Florig and Rivera (2002).

Theorem 2.1 If for all $i\in I, P_i$ is irreflexible and transitive and (survival assumption)

$$0\in coX_i-\{\omega_i\}-\sum_{j\in J}\theta_{ij}\lambda(T_j)\ coY_j$$

then for every economy ε *there exists a weak equilibrium with* $q\rangle 0$. *Moreover if* $m_i\rangle 0$ *for all* $i\in I$, *there exists a rationing equilibrium with* $q\rangle 0$.

We point out that the existence of equilibrium results holds under very weak assumptions on the economy.

3.- Core

In this section, we shall study the core properties of rationing equilibrium. In particular, we will establish a core equivalence result. To proceed, we begin with the following definition, which is a natural extension of standard weak core to our framework.

4 In that case, paper money can be used as médium of exchange. In the contrary case, due to paper money does not Participates in consumers preferences, it could drooped from the economy without further consequences.

Definition 3.1. A collection $(\chi, y) \in A\ (\varepsilon)$ is in the weak core if there does not exist $(\chi', y') \in X \times Y$ and the measurable set $T \subset I \text{ with } \lambda(T) > 0$ such that:

(i) for a.e. $t \in T, \chi'_t \in P_t(\chi_t)$;

(ii) $\int_T \chi'_t - e_t = \sum_{i\in I} \sum_{j\in J} \lambda(T \cap T_i)\theta_{ij}\lambda(T_j)y'_j$.

Proposition 3.1. Let χ, y, p, q be a weak equilibrium such that for all $i \in I, qm_i \rangle 0$, then $(\chi, y)$is in the weak core.

Proof. We proceed by contraposition

Let T, χ', y' as described in the definition. So for a.e. $t \in T$,

$$p \cdot \chi'_t \rangle p \cdot e_t + \sum_{j\in J} \theta_{tj}\pi_j(p) \geq p \cdot e_t + \sum_{j\in J} \theta_{tj}\lambda(T_j)p \cdot y'_j$$

Thus $p \cdot \int_T \chi'_t - e_t \rangle p \cdot \sum_{i\in I} \sum_{j\in J} \lambda(T \cap T_i)\theta_{ij}\lambda(T_j)y'_j$ contradicting *(ii)* of Definition 3.1.

Remark 3.1. The weak core cannot be decentralized.

Adapting an example from Konovalov (1998), consider an exchange economy with two types of consumers $(with \lambda(T_1) = \lambda(T_2))$ and two commodities. Let $X_1 = X_2 = \{0,1,2\}^2, u_1(\chi) = -\chi_1 + \chi_2, u_2(\chi) = \min\{\chi_1, \chi_2\}, e_1 = (2,0), e_2 = (0,2)$. The type-symmetric allocation $\chi_1 = (0,2), \chi_2 = (2,0)$ is in the weak core (in fact, it is even in the strong core, i.e. the one using weak blocking). By a demand characterization given in Florig and Rivera (2002), for all $p, q \in IR^L \times IR_+, \chi_2 \notin coD_2(p,q)$. So this allocation cannot be decentralized. One may check that we have a unique weak equilibrium allocation with $qm_i \rangle 0$ for all $i \in I$ which is in fact type symmetric: $\chi_1 = (0,0) \chi_2 = (2,2)$

Previous Remark lead us to consider a refinement of weak core, which is a natural extension of Konovalov (1998) definition to our setting.

Definition 3.2. *The coalition* $T \subset I$ **rejects** $(\chi, y) \in A\ (\varepsilon)$, *if there exist a measurable partition* U, V *of* T, *and an allocation* $\chi' \in X$ *such that the following holds:*

(i) $$\int_T x'_t \in \int_U \left[x_t + \sum_{j \in J} \theta_{tj} \int_{T_j} (Y_j - y_\tau) d\tau\right] dt + \int_V \left[e_t + \sum_{j \in J} \theta_{tj} \lambda(T_j) Y_j\right] dt;$$

(ii) *for a.e.* $t \in T, \chi'_t \in P_t(\chi_t)$

The **rejective core** $\mathcal{RC}(\mathcal{E})$ *of* ε *is the set of* $(\chi, y) \in A\ (\varepsilon)$ *which cannot by rejected by a non-negligible coalition.*

The interpretation of this core concept could be as follows. An allocation χ is proposed; group V refuses this allocation and stays with the initial endowment; group U realizes the proposed exchange and once they obtained the allocation χ, they meet with group V leading them to the allocation χ'.

Allocation χ' could be infeasible, if groups U and V were too big. However, one can always construct from U and V smaller groups U' and V' such that χ' is feasible for them. It is sufficient to choose them such that for all $i \in I, \lambda(U' \cap T_i) = \frac{1}{2}\lambda(U \cap T_i) and \lambda(V' \cap T_i) = \frac{1}{2}\lambda(V \cap T_i)$. Now if V' refuses to exchange, then a proportion larger than ½ of the set of agents can establish χ. The complement fails to establish χ since V' refused. They stay with their initial endowment. Then, U' and V' can indeed establish χ' together.

Remark 3.2. Rationing equilibria without money may be rejected.

Consider an exchange economy with three types of consumers $(with \lambda(T_1) = \lambda(T_2) = \lambda(T_3))$ and two commodities: for all $i \in I, X_i = \{0,1,2\}^2, u_1(\chi) = -\chi^1 - \chi^2, u_2(\chi) = -\|\chi - (1,1)\|_1, u_3(\chi) = -\|\chi - (0,1)\|_1, e_1 = (0,4), e_2 = (0,0), e_3 = (1,0)$ The type symmetric allocation $\chi_1 = (0,0), \chi_2 = (1,2), \chi_3 = (0,2)$ is a rationing equilibrium with $p = q = 0, K = \{t(0,-1) \mid t \geq 0\}$. However it is not in the rejective core since the players of type 2 and 3 may reject this leading them to $\xi_2 = (1,1)$ and $\xi_3 = (0,1)$ (type 2 agents accepts χ_2 and type 3 agents stay with their initial endowment).

Proposition 3.2. *Let (χ, y, p, q, K) be a rationing equilibrium such that for all $i \in I, qm_i \rangle 0$, then (χ, y) is in the rejective core.*

Proof. Let $T \subset I$ with $\lambda(T) \rangle 0$ and a measurable partition U, V of T and $\chi' \in X$ such that for a.e. $t \in T, \chi'_t \in P_t(\chi_t)$. Thus $\int_U \chi'_t - \chi_t \in K \setminus \{0\}$.

First note that

$$p \cdot \int_T \chi'_t = p \cdot \int_T e_t + q \int_T m_t + \sum_{i \in I} \lambda(T \cap T_i) \sum_{j \in J} \theta_{ij} \pi_j(p).$$

Thus, if condition (i) of Definition 5.2. is satisfied, we necessarily have $\lambda(V) = 0$.

Note that for every $j \in J, \int_{T_j} (Y_\tau - y_\tau) d\tau \subset -K.$ Thus

$$\int_U x'_t \in \int_T \left[x_t + \sum_{j \in J} \theta_{tj} \int_{T_j} (Y_j - y_\tau) d\tau \right] dt \subset \int_T x_t - K.$$

Thus, $\int_T \chi'_t - \chi_t \in -K$ and this contradicts $\int_T \chi'_t - \chi_t \in K \setminus \{0\}$.

The absence of some local non-satiation property would entail the existence of rejective core allocations which cannot be decentralized. This is due to the fact that a consumer at a satiation point does not care whether a firm he entirely owns chooses an efficient production plan or not (cf. Florig (2001)).

Proposition 3.3 *Suppose* $J = \theta$. *Then, for every* $x \in \mathcal{RC}(\mathcal{E})$ *there exists* $(p, m') \in I\!R^L \setminus \{0\} \times L^1(\mathcal{I}, I\!R_{++})$ *such that* $(\chi, p, q = 1)$ *is a Walras equilibrium with money of the economy* ε *when replacing* m *by* m'.

Proof. Let $x \in \mathcal{RC}(\mathcal{E})$. Since the number of types is finite and the consumption sets are finite, we can define a finite set of consumer types $A \equiv \{1,..., A\}$ satisfying the following:

(i) $(T_a)_{a \in A}$ is a finer partition of $\mathcal{I}$ than $(T_i)_{i \in I}$,

(ii) for every $a \in A$, there exists χ_a such that for every $t \in T_a, \chi_t = \chi_a$

Set

$$H_a = \lambda(T_a)(coP_a(\chi_a) - \chi_a), G_a = \lambda(T_a)(coP_a(\chi_a) - e_a),$$
$$\mathcal{K} = \mathrm{co} \cup_{a \in A} (G_a \cup H_a).$$

Claim 3.1. $0 \notin \mathcal{K}$.

Proof of Claim. Otherwise there exist $(\lambda_a), (\mu_a) \in [0,1]^A$ with $\sum_{a \in A} (\lambda_a + \mu_a) = 1$ and $\xi_a \in coP_a(\chi_a)$ for all $a \in A$ such that

$$\sum_{a \in A} [\lambda_a \lambda(T_a)(\xi_a - \chi_a) + \mu_a \lambda(T_a)(\xi_a - e_a)] = 0$$

Thus there exists $T \subset \mathcal{I}$, a measurable partition U, V of $T, \xi \in X$ such that for a.e. $t \in T, \xi_t \in P_t(\chi_t)$ and for all $a \in A, \lambda(U \cap T_a) = \lambda_a \lambda(T_a)$ and $\lambda(V \cap T_a) = \mu_a \lambda(T_a)$.

Thus $\int_T \xi_t = \int_U \chi_t + \int_V e_t$ contradicting $x \in \mathcal{RC}$.

Since $\mathcal{K}$ is compact, there exist $p \in IR^L \setminus \{0\}$ and $\varepsilon \rangle 0$ such that $\varepsilon \langle \min p \cdot \mathcal{K}$. For every $a \in A$, let $m'_a = p \cdot (\chi_a - e_a) + \varepsilon / 2$ and set $q = 1$. Then, of course for every $t \in \mathcal{I}$, $p \cdot x_t < p \cdot e_t + qm'_i < \min p \cdot P_t(x_t)$.

To end this section, we use an example from Shapley and Scarf (1974) to illustrate some facts mentioned in this section.

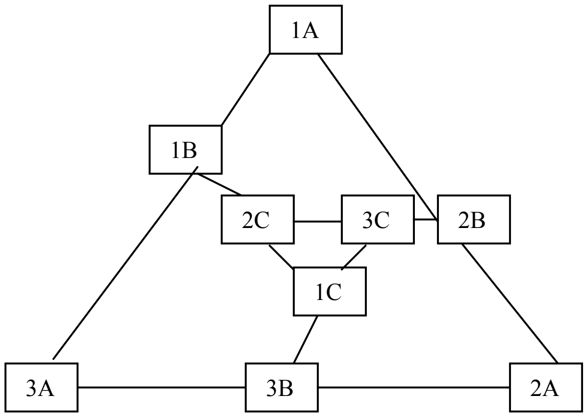
Example 3.1.

Shapley and Scarf (1974) gave the following example in order to show that the core may be empty when commodities are indivisible. We consider an economy with three types of agents $I = \{1,2,3\}$ nine commodities $L = \{1_A, 1_B, 1_{C,\ldots}, 3_C\}$ commodity set $X_i = \{0,1\}^9$ and concave utility functions for $i \in I$

$$u_i(\chi) = \max\{2\min\{\chi_{i_A}, \chi_i + 1_A, \chi_i + 1_B\}; \min\{\chi_{i_C}, \chi_{i+2_B}, \chi_{i+2_C}\}\}$$

The indices are module 3. Initial endowments are $e_i = (e_{ih}) \in X_i$ with $e_{ih} = 1$ if and only if $h \in \{i_A, i_B, i_C\}$.

The following picture illustrates endowments and preferences. Each consumer would like to have three commodities on a straight line containing only one of his commodities. The best bundle is to own a long line containing his commodity i_A and $i + 1_B, i + 1_A$ and the second best would be to own a short line containing his commodity i_C and $i + 2_B, i + 2_C$.



If there is only one agent per type this reduces indeed to Shapley and Scarf's (1974) setting. In this case, at any feasible allocation for some $i \in I$, agent i obtains utility zero and agent $i+2$ at most utility one. However, if they form a coalition it is possible to give utility one to i and two to $i+2$. Thus, the core is empty.

With an even number of agents per type or a continuum of measure one per type the weak and the rejective core correspond to the allocations such that half of the consumers of type i consume $\chi_{ih} = 1$ for all $h \in \{ i_A, i+1_A, i+1_B \}$ and the other half consumers $\chi_{ih} = 1$ for all $h \in \{ i_C, i+2_B, i+2_C \}$. So every consumer obtains at least his second best allocation. It is not possible to block an allocation in the sense that all consumers who block are better off. Indeed, they would all need to obtain their best allocation and this is not feasible for any group. To see that this is the only allocation in the core, note that at any other allocation at least one consumer say a consumer of type 1 (or a non-negligible group of a given type) would necessarily get an allocation which yields zero utility. Then by feasibility, a consumer of type 3 (or a non-negligible group of type 3) obtain only their second best choice. The consumer of type 1 can propose the commodities $1_A, 1_B$ in exchange for $3_B 3_C$ making everybody strickly better off.

Allocations in the core are supported by a uniform distribution of paper money $m_i = m \rangle 0$ for all $i \in I$ and the price $p = (2,1,1,2,1,1,2,1,1), q = 1/m$. Thus, a Walras equilibrium with money does not exist for a uniform distribution of paper money. A rationing equilibrium, however, exists. If half of each type obtains one unit of paper money and the other half strictly less than one unit, then the core allocation is a Walras equilibrium allocation with the same price p and $q = 1$.

4.- Welfare Analysis

We begin adapting the usual weak Pareto optimum definition to our framework and we shall prove that our equilibrium notion verifies welfare properties with this concept of Paretianity. Unfortunately, in our model of indivisibility strong Pareto optimality fails as we mentioned in the introduction.

Definition 4.1 *A collection* $(\chi, y) \in A(\varepsilon)$ *is a* **weak Pareto** *optimum if there does not exist* $(\chi', y') \in A(\varepsilon)$ *such that* $\chi'_t \in P_t(\chi_t)$ *for a.e.* $t \in \mathcal{I}$.

Remark 4.1. Weak equilibria with $q = 0$ need not be weak Pareto optima.

Consider an economy with two types of agents $I = \{1,2\}, L = \{A, B\}, X = \{0,1\}^2, e_1 = (1,0), e_2 = (0,1) u_1(\chi) = \chi_B, u_2(\chi) = \chi_A$. Then (χ, p, q) with $\chi_t = e_t$ for all $t, p = (1,1), q = 0$ is a weak equilibrium. However, type one agents consuming $(0,1)$ and type two agents $(1,0)$ increases the utility of all agents.

Following proposition comes readily from Proposition 3.1.

Proposition 4.1. First Welfare Theorem

a.- Every weak equilibrium (χ, y, p, q) *is weak Pareto optimum provides that* $qm_i \rangle 0$.

b.- Every rationing equilibrium is weak Pareto optimum.

In what follows, we restrict ourselves to exchange economies when studying the Second Welfare Theorem. Following Remark is illustrative of one difficulty we have to decentralize Pareto optimum with our equilibrium notion.

Remark 4.2. Weak Pareto optima cannot always be decentralized by $p \neq 0$[5].

Consider an exchange economy with three consumers and two commodities $L = \{A, B\}$: for all $i \in I, X_i = \{0,1,2\}^2, u_1(\chi) = 0, u_2(\chi) = \chi_A, u_3(\chi) = \chi_B, \chi_1 = (0,0), \chi_2 = (0,2), \chi_3 = (2,0)$. Decentralizing this allocation by $p \in IR^L \setminus \{0\}$ $\left(or(p,q) \in IR^L \times IR_+ with\ qm_i \rangle 0\right)$ implies that $p \in IR^L_{++}$. For $p_A \geq p_B, \chi \in D_2(p,q)$ implies $\chi_A \geq 1$ and for $p_A \leq p_B, \chi \in D_3(p,q)$. implies $\chi_B \geq 1$.

The problem in previous Remark comes from the fact that $P_i\chi$ could be an empty-set and therefore it would not possible to obtain a different from zero price which decentralize the point. However, if we consider a slightly weaker notion of Paretianity than the weak one, this problem could be dropped as we shall see in next proposition.

Definition 4.2. *A collection $(\chi, y) \in A\ (\varepsilon)$ is a* **feeble Pareto optimum** *if there does not exist $(\chi', y') \in A\ (\varepsilon)$ and a non-negligible set $T \subset \mathcal{I}$ such that for a.e. $t \in T, \chi'_t \in P_t(\chi_t)$ and for a.e. $t \in \mathcal{I},\ x'_t \neq x_t$ if and only if $t \in T$.*

Note that any feeble. Pareto optimum is a weak Pareto optimum.

Proposition 4.2. Second Welfare Theorem

Let ε be an economy with $J = \theta$. Let χ be a feeble Pareto optimum. Then there exists $p \in IR^L \setminus \{0\}$ and $e' \in X$ such that (χ, p) is a Walras equilibrium of ε' which is obtained from ε, replacing the initial endowment e by e'.

Proof. For all $t \in \mathcal{I}$ set $e'_t = x_t$. Since the number of types is finite and the consumption sets are finite, we can define a finite set of consumer types $A \equiv \{1, \ldots, A\}$ satisfying the following:

5 Or $(p,q) \in IR^L \times IR_+$ with $qm_i \rangle 0$

(i) $(T_a)_{a\in A}$ *is a finer partition of* $\mathcal{I}$ *than* $(T_i)_{i\in I}$,

(iii) *for every* $a \in A$, *there exists* χ_a *such that for every* $t \in T_a, \chi_t = \chi_a$.

Set $H_a = \lambda(T_a)(coP_a(\chi_a) - \chi_a)$ *and* $\mathcal{H} = \mathrm{co} \cup_{a\in A} H_a.$

Note that $0 \notin \mathcal{H}.$. Otherwise there exist $(\lambda_a) \in [0,1]^A$ with $\sum_{a\in A} \lambda_a = 1$ and $\xi_a \in coP_a(\chi_a)$ for all $a \in A$ such that $\sum_{a\in A} \lambda_a \lambda(T_a)(\xi_a - \chi_a) = 0$. Thus there exist $\xi \in X$ such that for all $a \in A, \lambda(\{\, t \in T_a | \xi_t \in P_t(\chi_t)\}) = \lambda_a \lambda(T_a)$ and $\lambda(\{t \in T_a | \xi_t = \chi_t\}) = (1-\lambda_a)\lambda(T_a)$ contradicting the weak Pareto optimality of χ.

As $\mathcal{H}$ is compact, there exists $p \in IR^L \setminus \{0\}$ and $\varepsilon \rangle 0$ such that for all $z \in \mathcal{H}, p\cdot z > \varepsilon$. Hence for a.e. $t \in \mathcal{I}, P_t(x_t) \cap \{\xi \in X_t \mid p\cdot\xi \le p\cdot x_t + \varepsilon\} = \emptyset.$. So (χ, p) is indeed a Walras equilibrium of ε'. Setting $q \rangle 0$ such that for all $i, qm_i \langle \varepsilon/2, (\chi, p, q)$ would also be a Walras equilibrium with a positive value of paper money.

Remark 4.3. Under the assumptions of the previous proposition, we could also decentralize any Pareto optimum χ by a bonafide fiscal policy. Collecting taxes $\tau_t = p\cdot(\chi_t - e_t) + m_t$ from agent $t \in \mathcal{I}$ payable in monetary units, χ becomes an equilibrium together with $q = 1$ and p as in the previous proof.

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