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Large Multi-Objective Generalized Games: Existence and Essential Stability of Equilibria

Autores:
Sofía Correa
Juan Pablo Torres-Martínez

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sdt@econ.uchile.cl
econ.uchile.cl/publicaciones

LARGE MULTI-OBJECTIVE GENERALIZED GAMES: EXISTENCE AND ESSENTIAL STABILITY OF EQUILIBRIA

SOFÍA CORREA AND JUAN PABLO TORRES-MARTÍNEZ

Abstract. We characterize the existence and the essential stability of Weak Pareto-Nash and Pareto-Nash equilibria in multi-objective generalized games with a continuum of players.

Keywords: Large Multi-Objective Generalized Games - Equilibria - Essential Stability
JEL Classification: C62, C72, C02.

1. Introduction

In many real-world situations, agents do not necessarily have the same objectives or intentions. There may be conflict of interests among agents as well as within an agent herself. The non cooperative character of a game represents the first conflict but, for including the second one as well, it should be incorporated the existence of a trade-off among tasks of a given player. This motivates the study of multi-objective games, that represent situations with political and management conflicts, where agents do not have a priori knowledge about the relative importance of the components of their payoffs.

This study is focused on large multi-objective generalized games, where there is a continuum of players and only a finite number of them can be atomic. For every player, the multi-objective function and the set of admissible strategies may depend on the atomic players' chosen strategies and on aggregated information about the non-atomic players' chosen strategies.

In this context, we obtain existence results for Weak Pareto-Nash and Pareto-Nash equilibria. In addition, several properties of essential stability are addressed, i.e., we analyze how equilibrium strategies change when some characteristics of the multi-objective game are perturbed. We allow any kind of perturbation on the characteristics of a multi-objective game, provided that it can be captured through a continuous parametrization. Hence, given a parametrization of the space of

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Sofía Correa
New York University
e-mail: sc4811@nyu.edu

Juan Pablo Torres-Martínez
Department of Economics, Faculty of Economics and Business, University of Chile
e-mail: juan.torres@fen.uchile.cl.

multi-objective games, we prove that Weak Pareto-Nash equilibria are generically essential. Moreover, we analyze the existence of essential subsets and components of the Weak Pareto-Nash equilibria collection. Furthermore, departing from stability results for Cournot-Nash equilibria in large generalized games, developed by Correa and Torres-Martínez (2014), we obtain essential stability properties for the set of Pareto-Nash equilibria of multi-objective games.

Since multi-objective games (and multi-objective generalized games) with finitely many players can be obtained as a particular case of our framework, we extend the previous literature on essential stability to address general types of admissible perturbations, developing a more comprehensive characterization of essential stability for multi-objective games.

In summary, we contribute to the game theory literature in several ways: (i) by obtaining existence of Pareto-Nash solutions in multi-objective games with infinitely many players; (ii) by allowing a broader range of admissible perturbations in the characteristics of multi-objective games; (iii) by proving that Weak Pareto-Nash equilibria are generically stable and that the uniqueness of equilibrium is a sufficient condition for essential stability; (iv) by discussing the existence of essential subsets and components of the set of Weak Pareto-Nash equilibria; (v) by characterizing the essential stability of Pareto-Nash equilibria.

The remaining part of this work is organized as follows. In Section 2 we discuss the related literature. Section 3 is devoted to describe the space of multi-objective generalized games. In Section 4 we prove the existence of Pareto-Nash equilibria, obtaining as a byproduct the existence of Weak Pareto-Nash equilibria. In Section 5 we develop the essential stability theory of Weak Pareto-Nash equilibria, allowing general types of perturbations in the characteristics of the multi-objective game. In Section 6 we complement these results with properties concerning the essential stability of Pareto-Nash solutions.

2. Related Literature

Multi-objective games have been initially studied by Blackwell (1956) and Shapley (1959), who analyze two-person zero-sum vector matrix games. More than thirty years later, Zhao (1991) and Wang (1991) develop the literature on existence of equilibria for multi-objective games with finitely many players. The first proves existence of several kinds of solution concepts (cooperative, non-cooperative and hybrids), while the second proves existence of Pareto equilibrium (i.e., allocations that induce Pareto optimal vectors of individual payoffs). In the context of multi-objective games with finitely many players, these results are extended to general topological spaces of strategies by Ding (2000), and to generalized multi-objective games by Kim and Ding (2003), Lin, Yang and Yu (2005), and Lin (2005). In the same context, Lin (2005) and Yu and Lin (2007) study the existence of Weak Pareto-Nash and Additive Weighted Nash equilibria.

Essential equilibrium stability has been developed to study the robustness of equilibrium solutions to perturbations in the characteristics of the game. This concept has its origins as a robustness property of fixed points of mappings (cf., Fort (1950), Kinochita (1952), Jia-He (1962, 1963), Yu and Yang (2004), Yu, Yang, and Xiang (2005)). Since in non-cooperative games the set of Nash equilibria coincides with the fixed points of a correspondence, we can depart from the mathematical analysis theory to discuss stability of solutions in games and generalized games (see, for instance, Wu and Jia-He (1962), Yu (1999), Yu, Yang, and Xiang (2005), Zhou, Yu and Xiang (2007), Yu (2009), Carbonell-Nicolau (2010), Correa and Torres-Martínez (2014)). Regarding multi-objective games with finitely many players, the first approach to essential stability was the analysis of the existence of stable (connected) components of the set of Weak Pareto-Nash equilibria (see Yang and Yu (2002), Lin, Yang and Yu (2005), and Lin (2005)). Also, previous results of the literature ensure that Weak Pareto-Nash equilibria are essential in a generic set of multi-objective generalized games with finitely many players (cf., Yu and Lin (2007), Song and Wang (2010)).

Our results complement this literature by extending the analysis of equilibrium existence and essential stability of Weak Pareto-Nash equilibria to generalized multi-objective games with a continuum of players, allowing general types of perturbations in the characteristics of the multi-objective games. Additionally, we contribute to the previous literature by including essential stability results for Pareto-Nash equilibria.

3. Large Multi-Objective Generalized Games

We analyze a type of large generalized games in which individuals may have multiple objectives. Let $((T, \mathcal{A}, \mu), (\widehat{K}_t)_{t\in T}, H)$ be the characteristics of multi-objective games that are not subject to perturbations, where $(T, \mathcal{A}, \mu)$ is a finite and complete measure space with a finite number of atoms. Let T_1 be the non-atomic part of T, which is assumed to be compact and metrizable. Non-atomic players' strategies are contained in a non-empty compact metric space $\widehat{K}_t \equiv \widehat{K}$, whose Borel σ-algebra is denoted by $\mathcal{B}(\widehat{K})$. Each atomic player $t \in T_2 := T \setminus T_1$ has strategies that are contained in a non-empty, convex, and compact set $\widehat{K}_t$, which is a subset of a normed vector space and it is equipped with the metric induced by the norm. The information contained in non-atomic players' strategies is codified by a function $H : T_1 \times \widehat{K} \to \mathbb{R}^m$, which is continuous with respect to the product topology induced by the metrics of T_1 and $\widehat{K}$.

In a large multi-objective generalized game $\mathcal{G}((K_t, \Gamma_t, U_t)_{t\in T})$, each player $t \in T$ has associated a strategy set $K_t \subseteq \widehat{K}_t$. A profile of strategies is given by a function $f : T \to \bigcup_{t\in T} \widehat{K}_t$ such that $f(t) \in K_t$ for each $t \in T$. Let $\mathcal{F}((K_t)_{t\in T})$ be the space of strategy profiles. For convenience of notation, define $\widehat{\mathcal{F}} = \mathcal{F}((\widehat{K}_t)_{t\in T})$.

Individuals cannot directly observe strategies of non-atomic players, but they can observe aggregated information about their decisions. More precisely, the relevant characteristics of non-atomic players' actions are coded by the function H, and each player takes into account aggregated information about these characteristics. Hence, given a strategy profile f, each player observes the message $m(f) := \left(\int_{T_1} H(t, f(t))d\mu,\, f|_{T_2}\right)$, where $f|_{T_2} := (f(t); t \in T_2)$.

Therefore, we define the set of messages that players can receive as follows:

$$M((K_t)_{t\in T}) = \left\{ \left(\int_{T_1} H(t, f(t))d\mu,\, f|_{T_2} \right) : \quad f \in \mathcal{F}((K_t)_{t\in T}) \;\; \wedge \;\; H(\cdot, f(\cdot)) \text{ is measurable} \right\}.$$

Let $\widehat{M} = M((\widehat{K}_t)_{t\in T})$. Admissible strategies can be restricted by messages, i.e., the set of strategies available for a player $t \in T$ is determined by a correspondence $\Gamma_t : \widehat{M} \twoheadrightarrow K_t$. Finally, the multi-tasks of a player $t \in T$ are given by a vector valued function $U_t : \widehat{K}_t \times \widehat{M} \to \mathbb{R}^n$ where

$U_t = (u_{t,1}, \dots, u_{t,n})$.[1]

Definition 1 (Weak Pareto-Nash Equilibrium)
A Weak Pareto-Nash equilibrium of $\mathcal{G}((K_t, \Gamma_t, U_t)_{t\in T})$ *is given by a strategy profile* $f^* \in \widehat{\mathcal{F}}$ *such that, for almost all player* $t \in T$, $f^*(t) \in \Gamma_t(m(f^*))$ *and there is no* $f(t) \in \Gamma_t(m(f^*))$ *satisfying* $U_t(f(t), m(f^*)) - U_t(f^*(t), m(f^*)) \in \mathbb{R}^n_{++}$.

Definition 2 (Pareto-Nash Equilibrium)
A Pareto-Nash equilibrium of $\mathcal{G}((K_t, \Gamma_t, U_t)_{t\in T})$ *is given by a strategy profile* $f^* \in \widehat{\mathcal{F}}$ *such that, for almost all player* $t \in T$, $f^*(t) \in \Gamma_t(m(f^*))$ *and there is no* $f(t) \in \Gamma_t(m(f^*))$ *satisfying* $U_t(f(t), m(f^*)) - U_t(f^*(t), m(f^*)) \in \mathbb{R}^n_{+} \setminus \{0\}$.

Note that, the set of Weak Pareto-Nash equilibria of a large multi-objective generalized game $\mathcal{G} := \mathcal{G}((K_t, \Gamma_t, U_t)_{t\in T})$, denoted by $\mathcal{WPN}(\mathcal{G})$, contains the set of Pareto-Nash equilibria $\mathcal{PN}(\mathcal{G})$. The following example shows that these sets do not necessarily coincide.

Example 1. Consider the multi-objective generalized game $\mathcal{G}_\epsilon$ characterized by $T = \{1, 2\}$, $\widehat{K}_t = K_t = \Gamma_t(\cdot) = [0, 1] \times [0, 1]$, $\forall t \in T$. Objective functions are $U_1^\epsilon(x_1, x_2) = (x_1(1 - x_1), x_2 - \epsilon x_1)$ and $U_2(y_1, y_2) = -\left(\|(y_1, y_2) - (x_1, x_2)\|, \|(y_1, y_2) - (x_1, x_2)\|\right)$, where $\epsilon \in (0, 1)$ is fixed.

Then, the set of weak Pareto-Nash equilibria of $\mathcal{G}_\epsilon$ is given by

$$\mathcal{WPN}(\mathcal{G}_\epsilon) = \left\{ \left(\left(\frac{1}{2}, \lambda\right), \left(\frac{1}{2}, \lambda\right) \right) : \ \lambda \in [0, 1) \right\} \bigcup \left\{ \left((\lambda, 1), (\lambda, 1) \right) : \ \lambda \in \left[0, \frac{1}{2}\right] \right\},$$

while $\mathcal{PN}(\mathcal{G}_\epsilon) = \left\{ ((\lambda, 1), (\lambda, 1)) \in [0, 1]^2 \times [0, 1]^2 : \lambda \in \left[0, \frac{1}{2}\right] \right\}$. □

4. Existence of Equilibria

We show the existence of Pareto-Nash equilibria by transforming each large multi-objective generalized game into a large generalized game, where each agent changes her multi-objective function by a weighted average of her tasks. Thus, we can apply the result of equilibrium existence for large generalized games developed by Balder (2002).

Given $((T, \mathcal{A}, \mu), (\widehat{K}_t)_{t\in T}, H)$, the following assumptions summarize the requirements over a large multi-objective game $\mathcal{G}((K_t, \Gamma_t, U_t)_{t\in T})$ to ensure the existence of Pareto-Nash equilibria.

(A1) For any player $t \in T$, K_t is non-empty and closed, U_t is continuous in the sup-norm topology, and Γ_t is continuous and has nonempty and compact values.

(A2) For any $t \in T_2$, K_t is convex, U_t is concave in its own strategy, and Γ_t has convex values.

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[1] We assume that all players have the same number of tasks, which is equivalent to supposing that: (i) any $t \in T$ has n_t tasks, i.e. $U_t = (u_{t,1}, \dots, u_{t,n_t})$, and (ii) $\{n_t : t \in T\} \subseteq \mathbb{N}$ is bounded. Indeed, under (ii) we can repeat some of the tasks $\{u_{t,1}, \dots, u_{t,n_t}\}$ to those agents $t \in T$ for which $n_t < \max_{k\in T} n_k$ increasing the number of objectives to $\max_{k\in T} n_k$, without changing the meaning of our equilibrium concepts.

(A3) For every $m \in \widehat{M}$, the correspondence $t \in T_1 \twoheadrightarrow \Gamma_t(m)$ is measurable.

(A4) For every $m \in \widehat{M}$, the function $(t,x) \in T_1 \times \widehat{K} \to U_t(x,m)$ is $\mathcal{A} \times \mathcal{B}(\widehat{K})$-measurable.

(A5) The map $(t,x,m) \in T_1 \times \widehat{K} \times \widehat{M} \to U_t(x,m)$ is bounded.

Theorem 1. *Let $\mathcal{G}((K_t,\Gamma_t,U_t)_{t\in T})$ be a large multi-objective generalized game. Under Assumptions (A1)-(A5), the set of Pareto-Nash equilibria is nonempty.*

Proof. Given $t \in T$, define $V_t : \widehat{K}_t \times \widehat{M} \to \mathbb{R}$ by $V_t(x,m) = \sum_{i=1}^n u_{t,i}(x,m)$. Under Assumptions (A1)-(A5), the hypotheses of the equilibrium existence theorem of Balder (2002, Theorem 2.2.1) hold (see also Correa and Torres-Martínez (2014, footnote 7)). Therefore, the set of Cournot-Nash equilibria of the large generalized game $\mathcal{G}((K_t,\Gamma_t,V_t)_{t\in T})$ is non-empty. Since every Cournot-Nash equilibrium of $\mathcal{G}((K_t,\Gamma_t,V_t)_{t\in T})$ is a Pareto-Nash equilibrium of $\mathcal{G}((K_t,\Gamma_t,U_t)_{t\in T})$, the proof is concluded. □

Corollary. *Let $\mathcal{G}((K_t,\Gamma_t,U_t)_{t\in T})$ be a large multi-objective generalized game. Under Assumptions (A1)-(A5), the set of Weak Pareto-Nash equilibria is nonempty.*

Let $\mathbb{G} := \mathbb{G}((T,\mathcal{A},\mu),(\widehat{K}_t)_{t\in T},H)$ be the collection of large multi-objective generalized games satisfying Assumptions (A1)-(A5). Since the analysis of essential stability is based on the idea that small perturbations in the characteristics of a game induce small perturbations in the set of equilibria, we need a way to measure the distance between multi-objective games. Thus, given $\mathcal{G}_1((K_t^1,\Gamma_t^1,U_t^1)_{t\in T})$ and $\mathcal{G}_2((K_t^2,\Gamma_t^2,U_t^2)_{t\in T})$ in $\mathbb{G}$, we consider the *uniform metric*

$$\rho(\mathcal{G}_1,\mathcal{G}_2) = \sup_{(t,m)\in T\times\widehat{M}} \left(\max_{x\in\widehat{K}_t} \sum_{i=1}^n |u_{t,i}^1(x,m) - u_{t,i}^2(x,m)| \,+\, d_H(\Gamma_t^1(m),\Gamma_t^2(m)) \,+\, d_H(K_t^1,K_t^2) \right),$$

where d_H is the Hausdorff metric induced by the spaces $(\widehat{K}_t)_{t\in T}$. Since $(T_1,(\widehat{K}_t)_{t\in T})$ are compact sets, T_2 is finite, and $\widehat{M}$ is compact, it follows that $(\mathbb{G},\rho)$ is a complete metric space (for additional details, see Correa and Torres-Martínez (2014, Proposition 1 and Lemma 1)).

5. Essential Stability of Weak Pareto-Nash Equilibria

In this section, we analyze the robustness of Weak Pareto-Nash equilibria regarding perturbations in the parameters that define the multi-objective game. Since multi-objective generalized games with *finitely many players* are particular cases of our framework, our results also complement the previous literature by allowing different players to suffer perturbations in different types of characteristics (payoffs, strategy sets, or admissible strategies).[2]

We study the stability of Weak Pareto-Nash equilibria in terms of parameterizations of the complete metric space $(\mathbb{G}_{\mathcal{W}},\rho)$, where $\mathbb{G}_{\mathcal{W}} \subset \mathbb{G}$ is the set of multi-objective games satisfying:

[2]Previous results of essential stability for multi-objective games with finitely many players analyze how the set of Weak Pareto-Nash equilibria changes when payoffs functions are perturbed (see, for instance, Yang and Yu (2002), Lin (2005), Lin, Yang and Yu (2005), Yu and Lin (2007), and Song and Wang (2010)). However, it is natural to discuss essential stability allowing any kind of perturbations of the characteristics of the generalized game.

(A6) For each $t \in T_1$, the correspondence Γ_t has convex values.

(A7) For each $t \in T_1$ and $m \in \widehat{M}$, the function $U_t(\cdot, m) : \widehat{K} \to \mathbb{R}^n$ is $\mathbb{R}^n_+$-quasiconcave.[3]

A *parametrization* $\mathcal{T} = ((\mathbb{X}, \tau), \kappa)$ of the space $\mathbb{G}_{\mathcal{W}}$ is given by a complete metric space of parameters $(\mathbb{X}, \tau)$ and a continuous function $\kappa : \mathbb{X} \to \mathbb{G}_{\mathcal{W}}$. For instance, if we consider $(\mathbb{X}, \tau) = (\mathbb{G}_{\mathcal{W}}, \rho)$ and $\kappa(\mathcal{G}) = \mathcal{G}$, then any perturbation in the characteristics of a multi-objective game is allowed. Alternatively, if we take $(K_t, \Gamma_t)_{t\in T}$ as given, such that they satisfy (A1)-(A3) and (A7), then we can focus on objective functions' perturbations by defining $\mathbb{X} = \{(U_t)_{t\in T} : \text{Assumption (A1)-(A7) hold}\}$, $\tau = \rho$, and $\kappa((U_t)_{t\in T}) = \mathcal{G}((K_t, \Gamma_t, U_t)_{t\in T})$.

Let $\Psi_{\mathcal{W}} : \mathbb{G}_{\mathcal{W}} \twoheadrightarrow \widehat{M}$ be the *Weak Pareto-Nash correspondence*, which associates to each multi-objective game $\mathcal{G} \in \mathbb{G}_{\mathcal{W}}$ the set of messages associated to its Weak Pareto-Nash equilibria, i.e., $\Psi_{\mathcal{W}}(\mathcal{G}) = \{m \in \widehat{M} : \exists f \in \mathcal{WPN}(\mathcal{G}),\ m = m(f)\}$. Note that, given $f \in \mathcal{WPN}(\mathcal{G})$, the message $m(f)$ contains all the information that players take into account to decide their optimal actions.

Definition 3 ($\mathcal{WPN}$-Essential Equilibrium)
Given a parametrization $\mathcal{T} = ((\mathbb{X}, \tau), \kappa)$ of $\mathbb{G}_{\mathcal{W}}$, a strategy profile $f^* \in \mathcal{F}$ is a $\mathcal{WPN}$-essential equilibrium with respect to $\mathcal{T}$ if there exists $X \in \mathbb{X}$ such that $f^* \in \mathcal{WPN}(\kappa(\mathcal{X}))$, and for any open neighborhood O of the message $m(f^*)$ there exists $\delta > 0$ such that, $\Psi_{\mathcal{W}}(\kappa(X')) \cap O \neq \emptyset$, for every $X' \in \mathbb{X} : \rho(X, X') < \delta$.

Definition 4 ($\mathcal{WPN}$-Essential Multi-Objective Game)
Given a parametrization $\mathcal{T} = ((\mathbb{X}, \tau), \kappa)$ of $\mathbb{G}_{\mathcal{W}}$, a multi-objective large generalized game $\mathcal{G} \in \mathbb{G}_{\mathcal{W}}$ is $\mathcal{WPN}$-essential with respect to $\mathcal{T}$ if there exists $X \in \mathbb{X}$ such that $\mathcal{G} = \kappa(X)$, and all its Weak Pareto-Nash equilibria are $\mathcal{WPN}$-essential with respect to $\mathcal{T}$.[4]

Unfortunately, as the following example shows, not all large multi-objective generalized games are $\mathcal{WPN}$-essential with respect to a parametrization.

Example 2. Given $\epsilon \in [0, 1]$, let $\mathcal{G}_\epsilon$ be the multi-objective game defined in Example 1 and consider the parametrization $\mathcal{T} = (([0, 1], |\cdot|), \kappa)$, where $\kappa : [0, 1] \to \mathbb{G}$ is given by $\kappa(\epsilon) = \mathcal{G}_\epsilon$. Note that, as there are only atomic players, for any $\epsilon \in [0, 1]$ the set of Weak Pareto-Nash equilibria coincides with the set of equilibrium messages, i.e., $\mathcal{WPN}(\mathcal{G}_\epsilon) = \Psi_{\mathcal{W}}(\mathcal{G}_\epsilon)$.

We affirm that $\kappa(0)$ in not $\mathcal{WPN}$-essential with respect to $\mathcal{T}$. Indeed, it follows from Example 1 that, for each $\epsilon > 0$, $\Psi_{\mathcal{W}}(\mathcal{G}_\epsilon) = \left\{\left(\left(\frac{1}{2}, \lambda\right), \left(\frac{1}{2}, \lambda\right)\right) : \lambda \in [0, 1)\right\} \bigcup \left\{\left((\lambda, 1), (\lambda, 1)\right), : \lambda \in \left[0, \frac{1}{2}\right]\right\}.$

Since, $\Psi_{\mathcal{W}}(\mathcal{G}_0) = \left\{\left(\left(\frac{1}{2}, \lambda\right), \left(\frac{1}{2}, \lambda\right)\right) : \lambda \in [0, 1)\right\} \bigcup \left\{\left((\lambda, 1), (\lambda, 1)\right) : \lambda \in [0, 1]\right\}$, we conclude that there is no element in $\Psi_{\mathcal{W}}(\mathcal{G}_\epsilon)$ sufficiently close to $((1, 1), (1, 1)) \in \Psi_{\mathcal{W}}(\mathcal{G}_0)$. □

[3]A mapping $g : \widehat{K} \to \mathbb{R}^n$ is $\mathbb{R}^n_+$-quasiconcave if and only if, for any $(x_1, x_2) \in \widehat{K} \times \widehat{K}$ and each $\lambda \in [0, 1]$, $g(\lambda x_1 + (1 - \lambda)x_2) \geq g(x_j)$ for some $j \in \{1, 2\}$.

[4]Note that, we make explicit the dependence of the stability property on the solution concept of equilibrium.

Before obtaining results about essential stability for Weak Pareto-Nash equilibria, we state some technical properties of the Weak Pareto-Nash correspondence $\Psi_{\mathcal{W}} : \mathbb{G}_{\mathcal{W}} \twoheadrightarrow \widehat{M}$.

PROPOSITION 1. *For any $\mathcal{G} \in \mathbb{G}_{\mathcal{W}}$, $\Psi_{\mathcal{W}}(\mathcal{G})$ coincides with the fixed points of a closed-graph correspondence $\Phi_{\mathcal{G}} : \widehat{M} \twoheadrightarrow \widehat{M}$. In particular, $\Psi_{\mathcal{W}}(\mathcal{G})$ is compact.*

PROPOSITION 2. *The correspondence $\Psi_{\mathcal{W}} : \mathbb{G}_{\mathcal{W}} \twoheadrightarrow \widehat{M}$ is upper hemi-continuous and there is a dense residual subset of $\mathbb{G}_{\mathcal{W}}$ where it is continuous.*[5]

The following result guarantees that essential stability is a generic property.

THEOREM 2. *Let $\mathcal{T} = ((\mathbb{X}, \tau), \kappa)$ be a parametrization of $\mathbb{G}_{\mathcal{W}}$.*
(i) There is a dense residual subset $\mathbb{X}' \subseteq \mathbb{X}$ such that, for all $X \in \mathbb{X}'$, the large multi-objective generalized game $\kappa(X)$ is $\mathcal{WPN}$-essential with respect to $\mathcal{T}$.
(ii) Given $X \in \mathbb{X}$, if $\Psi_{\mathcal{W}}(\kappa(X))$ is a singleton, then $\kappa(X)$ is $\mathcal{WPN}$-essential with respect to $\mathcal{T}$.

PROOF. We know that $\Psi_{\mathcal{W}} : \mathbb{G}_{\mathcal{W}} \twoheadrightarrow \widehat{M}$ is upper hemi-continuous and has non-empty and compact values. Since $\kappa : \mathbb{X} \to \mathbb{G}_{\mathcal{W}}$ is continuous, the composed correspondence $\Psi_{\mathcal{W}} \circ \kappa : \mathbb{X} \twoheadrightarrow \widehat{M}$ is closed and has non-empty and compact values. The completeness of $(\mathbb{X}, \tau)$ and Lemma 6 in Carbonell-Nicolau (2010) guarantee that there exists a dense residual subset $\mathbb{X}' \subseteq \mathbb{X}$ in which $\Psi_{\mathcal{W}} \circ \kappa$ is lower hemi-continuous. Thus, for any $X \in \mathbb{X}'$, $\kappa(X)$ is $\mathcal{WPN}$-essential with respect to $\mathcal{T}$. Finally, if $\Psi_{\mathcal{W}}$ is upper hemi-continuous and single valued at $\kappa(X)$, then it is lower hemi-continuous at this point. □

DEFINITION 5 ($\mathcal{WPN}$-ESSENTIAL SET)
Given a parametrization $\mathcal{T} = ((\mathbb{X}, \tau), \kappa)$ of $\mathbb{G}_{\mathcal{W}}$ and $X \in \mathbb{X}$, a set $E \subseteq \Psi_{\mathcal{W}}(\kappa(X))$ is $\mathcal{WPN}$-essential with respect to $\mathcal{T}$ if it is non-empty, compact, and for each open set $O \subseteq \widehat{M}$ with $E \subseteq O$ there exists $\delta > 0$ such that $\Psi_{\mathcal{W}}(\kappa(X')) \cap O \neq \emptyset$, for every $X' \in \mathbb{X} : \rho(X, X') < \delta$.

Let $E \subseteq \Psi_{\mathcal{W}}(\kappa(X))$ be a $\mathcal{WPN}$-essential set with respect to a parametrization $\mathcal{T}$. E is a *minimal $\mathcal{WPN}$-essential set* with respect to $\mathcal{T}$ if there is no $F \subset E$ that is $\mathcal{WPN}$-essential with respect to $\mathcal{T}$. In addition, E is a *$\mathcal{WPN}$-essential component* with respect to $\mathcal{T}$ if there exists $m \in \Psi_{\mathcal{W}}(\kappa(X))$ such that, E is the union of all connected subsets of $\Psi_{\mathcal{W}}(\kappa(X))$ containing m.

THEOREM 3. *Given a parametrization $\mathcal{T} = ((\mathbb{X}, \tau), \kappa)$ of $\mathbb{G}_{\mathcal{W}}$ and $X \in \mathbb{X}$, we have that:*
(i) $\Psi_{\mathcal{W}}(\kappa(X))$ has at least one minimal $\mathcal{WPN}$-essential subset with respect to $\mathcal{T}$.
(ii) If $\Psi_{\mathcal{W}}(\kappa(X))$ has a connected $\mathcal{WPN}$-essential set, then it has a $\mathcal{WPN}$-essential component.

PROOF. (i) Let $\mathcal{S}$ be the family of $\mathcal{WPN}$-essential sets of $\Psi_{\mathcal{W}}(\kappa(X))$ with respect to $\mathcal{T}$ ordered by set inclusion. Since $\Psi_{\mathcal{W}}$ is upper-hemicontinuous and has compact values, $\Psi_{\mathcal{W}}(\kappa(X)) \in \mathcal{S}$. As any element of $\mathcal{S}$ is compact, any totally ordered subset of $\mathcal{S}$ has a lower bounded element. By Zorn's Lemma, $\mathcal{S}$ has a minimal element.

[5] $\mathbb{G}' \subseteq \mathbb{G}$ is *residual* if it contains the intersection of a countable collection of dense and open subsets of $\mathbb{G}$.

(ii) Let $Z_\kappa(X)$ be a connected $\mathcal{WPN}$-essential set of $\Psi_{\mathcal{W}}(\kappa(X))$ with respect to $\mathcal{T}$ and fix $m \in Z_\kappa(X)$. Let C_m be the union of all connected subsets of $\Psi_{\mathcal{W}}(\kappa(X))$ containing m. By definition, C_m is a component of $\Psi_{\mathcal{W}}(\kappa(X))$. Since it is compact, and contains $Z_\kappa(X)$, it is also $\mathcal{WPN}$-essential with respect to the parametrization $\mathcal{T}$. □

Note that, Theorem 2(ii) ensures that the uniqueness of Weak Pareto-Nash equilibria is a sufficient condition for essential stability. Under additional restrictions on non-atomic players' action spaces, the following result ensures that any large multi-objective generalized game with a finite set of Weak Pareto-Nash equilibrium messages has at least one essential equilibrium.

THEOREM 4. *Assume that $\widehat{K}$ is a convex subset of a normed linear space and its metric is induced by the norm. Given a parametrization $\mathcal{T} = ((\mathbb{X}, \tau), \kappa)$ of $\mathbb{G}_{\mathcal{W}}$ and $X \in \mathbb{X}$, we have that:*
(i) Every minimal $\mathcal{WPN}$-essential set with respect to $\mathcal{T}$ is connected.
(ii) If the set of equilibrium messages $\Psi_{\mathcal{W}}(\kappa(X))$ is finite, then at least one Weak Pareto-Nash equilibrium of $\kappa(X)$ is $\mathcal{WPN}$-essential with respect to $\mathcal{T}$.

PROOF. (i) It follows from Correa and Torres-Martínez (2014, Theorem 2, item (iii)).
(ii) If $\Psi_{\mathcal{W}}(\kappa(X))$ is finite, then it follows from Theorem 3(i) and the previous item that $\Psi_{\mathcal{W}}(\kappa(X))$ has at least one minimal and connected $\mathcal{WPN}$-essential subset with respect to $\mathcal{T}$. Thus, there is at least one Weak Pareto-Nash equilibrium of $\kappa(X)$ that is $\mathcal{WPN}$-essential with respect to $\mathcal{T}$. □

6. Essential Stability of Pareto-Nash Equilibria

Since the set of Pareto-Nash equilibria of a large multi-objective generalized game is contained in the set of Weak Pareto-Nash equilibria, up to now we ensure that Pareto-Nash equilibria of a multi-objective game can be generically approximated by Weak Pareto-Nash equilibria of nearby multi-objective games. However, since the set of Weak Pareto Nash equilibria and the set of Pareto-Nash equilibria do not necessarily coincide, it is interesting to analyze essential stability for each equilibrium concept.

Unfortunately, we cannot directly analyze the essential stability of Pareto-Nash equilibria. Indeed, to guarantee the results about essential stability, it is crucial to ensure that the correspondence that associates multi-objective games with equilibrium messages is upper hemi-continuous and has compact values. These properties do not necessarily hold for Pareto-Nash equilibria, as the following example shows.

EXAMPLE 3. Given $\epsilon \in [0,1]$, and following Example 1, let $\widetilde{\mathcal{G}}_\epsilon$ be the multi-objective game obtained from $\mathcal{G}_\epsilon$ by changing U_1^ϵ to $\widetilde{U}_1^\epsilon(x_1,x_2) = (x_1(1-x_1), x_2 - \min\{x_1;0.5\} + \epsilon \max\{x_1;0.5\})$.

Hence, the set of Pareto-Nash equilibria of $\widetilde{\mathcal{G}}_0$ is given by $\mathcal{PN}(\widetilde{\mathcal{G}}_0) = \left\{((\lambda,1),(\lambda,1)) : \lambda \in \left[0,\frac{1}{2}\right]\right\}$. In addition, for any $\epsilon > 0$, $\mathcal{PN}(\widetilde{\mathcal{G}}_\epsilon) = \{((\lambda,1),(\lambda,1)) : \lambda \in [0,1]\}$.

Let $\Psi_{\mathcal{P}} : \mathbb{G} \twoheadrightarrow \widehat{M}$ be the correspondence that associates any multi-objective game $\mathcal{G} \in \mathbb{G}$ its set of Pareto-Nash equilibrium messages. Note that, $\Psi_{\mathcal{P}}(\widetilde{\mathcal{G}}_\epsilon) = \mathcal{PN}(\widetilde{\mathcal{G}}_\epsilon)$, for all $\epsilon \in [0,1]$. It follows that, $\left\{\left(\widetilde{\mathcal{G}}_{\frac{1}{n}}, ((1,1),(1,1))\right)\right\}_{n\in\mathbb{N}}$ is contained in the graph of $\Psi_{\mathcal{P}}$ and converges to $\left(\widetilde{\mathcal{G}}_0, ((1,1),(1,1))\right)$. Since $((1,1),(1,1)) \notin \Psi_{\mathcal{P}}(\widetilde{\mathcal{G}}_0)$, it follows that $\Psi_{\mathcal{P}}$ does not have a closed graph. Therefore, $\Psi_{\mathcal{P}}$ is not upper hemi-continuous with compact values. □

Given these limitations, we focus on obtaining stability properties for a more restrictive solution concept for multi-objective games, that we refer as F-Weighted Cornot-Nash equilibrium, which assumes that players can aggregate tasks before making decisions, using a function $F : T \times \mathbb{R}^n \to \mathbb{R}$. Thus, we are able to adapt the results of essential stability for large generalized games of Correa and Torres-Martínez (2014) to deduce results of essential stability for Pareto-Nash equilibria of multi-objective generalized games.

Definition 6 (F-Weighted Cournot-Nash Equilibrium)
Given a map $F : T \times \mathbb{R}^n \to \mathbb{R}$, an F-Weighted Cournot-Nash equilibrium of the multi-objective game $\mathcal{G}((K_t, \Gamma_t, U_t)_{t\in T}) \in \mathbb{G}$ is given by a strategy profile $f^* \in \widehat{\mathcal{F}}$ such that, for almost all player $t \in T$, $f^*(t) \in \Gamma_t(m(f^*))$ and $F(t, U_t(f^*(t), m(f^*))) \geq F(t, U_t(f(t), m(f^*)))$, $\forall\, f(t) \in \Gamma_t(m(f^*))$.

By definition, the set of F-Weighted Cournot-Nash equilibria of $\mathcal{G}((K_t, \Gamma_t, U_t)_{t\in T}) \in \mathbb{G}$ coincides with the set of Cournot-Nash equilibria of the large generalized game $\mathcal{G}((K_t, \Gamma_t, (F(t, U_t(\cdot)))_{t\in T})$.[6]

Therefore, if we assume that $F : T \times \mathbb{R}^n \to \mathbb{R}$ is continuous, it follows from Balder (2002) and Correa and Torres-Martínez (2014), that any multi-objective game in $\mathbb{G}$—which by definition satisfies Assumptions (A1)-(A5)—has a non-empty set of F-Weighted Cournot-Nash equilibria.

Let $\Psi_F : \mathbb{G} \twoheadrightarrow \widehat{M}$ be the *F-Weighted Cournot-Nash correspondence*, which carry out any $\mathcal{G} \in \mathbb{G}$ to the set of messages associated to its F-Weighted Cournot-Nash equilibria.

Given a parametrization $\mathcal{T} = ((\mathbb{X}, \tau), \kappa)$ of $\mathbb{G}$, changing the set of Weak Pareto-Nash equilibria in Definitions 3-5 by the set of F-Weighted Cournot-Nash equilibria, and replacing $\Psi_{\mathcal{W}}$ by Ψ_F, we obtain notions of stability for the F-Weighted Cournot-Nash solution concept, that we refer as F-essential equilibrium, F-essential multi-objective game, and F-essential set (with respect to $\mathcal{T}$).

The following results characterize properties of essential stability for F-Weighted Cournot-Nash Equilibria using previous findings of Correa and Torres-Martínez (2014) for large generalized games. Since Assumptions (A6)-(A7) are not required in the theory of essential stability for large generalized games, we can work with parametrizations of $\mathbb{G}$.

Proposition 3. *Let $\mathcal{T} = ((\mathbb{X}, \tau), \kappa)$ be a parametrization of $\mathbb{G}$. Given $F : T \times \mathbb{R}^n \to \mathbb{R}$ continuous, there is a dense residual set $\mathbb{X}' \subseteq \mathbb{X}$ such that, for all $X \in \mathbb{X}'$, the large multi-objective generalized game $\kappa(X)$ is F-essential with respect to $\mathcal{T}$.*
Furthermore, for every parameter $X \in \mathbb{X}$ we have that:
(i) If $\Psi_F(\kappa(X))$ is a singleton, then $\kappa(X)$ is F-essential with respect to $\mathcal{T}$.
(ii) $\Psi_F(\kappa(X))$ has at least one minimal F-essential subset with respect to $\mathcal{T}$.
(iii) If $\Psi_F(\kappa(X))$ has a connected F-essential set, then it has a F-essential component.

[6] The F-Weighted Cournot-Nash equilibria is an extension to large multi-objective generalized games of the concept of Weight Nash equilibrium introduced by Kim and Ding (2003) for multi-objective generalized games with finitely many players (see also Yu and Lin (2007)).

Proof. For any $X \in \mathbb{X}$, let $(K_t^X, \Gamma_t^X, U_t^X)_{t\in T}$ be players' characteristics in $\kappa(X) \in \mathbb{G}$. Consider the parametrization of the space of continuous large generalized *games*, $\widetilde{\mathcal{T}} = ((\mathbb{X},\tau),\widetilde{\kappa})$, where for every parameter X the characteristics of players in $\widetilde{\kappa}(X)$ are given by $(K_t^X, \Gamma_t^X, F(t, U_t^X(\cdot))\,)_{t\in T}$.

Since, for every $X \in \mathbb{X}$, the set of F-Weighted Cournot-Nash equilibria of $\kappa(X)$ coincides with the set of Cournot-Nash equilibria of $\widetilde{\kappa}(X)$, $\kappa(X)$ is F-essential with respect to $\mathcal{T}$ if and only if $\widetilde{\kappa}(X)$ is $\widetilde{\mathcal{T}}$-essential in the sense of Correa and Torres-Martínez (2014, Definition 6).

Therefore, the results follow from Theorem 3(i)-(iii) in Correa and Torres-Martínez (2014). □

Proposition 4. *Assume that $\widehat{K}$ is a convex subset of a normed linear space and its metric is induced by the norm. Given a parametrization $\mathcal{T} = ((\mathbb{X},\tau),\kappa)$ of $\mathbb{G}$ and a continuous function $F : T \times \mathbb{R}^n \to \mathbb{R}$, for any $X \in \mathbb{X}$ we have that:*
(i) Every minimal F-essential subset of $\Psi_F(\kappa(X))$ with respect to $\mathcal{T}$ is connected.
(ii) If the set of equilibrium messages $\Psi_F(\kappa(X))$ is finite, then at least one F-Weighted Cournot-Nash equilibrium of $\kappa(X)$ is F-essential with respect to $\mathcal{T}$.

Proof. By the same arguments made in the proof of Proposition 3, this result is a consequence of Theorem 2(iii) and Theorem 3(iv) of Correa and Torres-Martínez (2014). □

Let $\Psi_{\mathcal{P}} : \mathbb{G} \twoheadrightarrow \widehat{M}$ be the *Pareto-Nash correspondence*, i.e., the set value function that associates to any $\mathcal{G} \in \mathbb{G}$ the set of Pareto-Nash equilibrium messages. The concepts of essential stability for Pareto-Nash equilibria—that we refer as $\mathcal{PN}$-essential equilibria, $\mathcal{PN}$-essential multi-objective game, and $\mathcal{PN}$-essential set—can be obtained from Definitions 3-5 by changing the set of Weak Pareto-Nash equilibria by the set of Pareto-Nash equilibria, and replacing $\Psi_{\mathcal{W}}$ by $\Psi_{\mathcal{P}}$.

Consider a function $F : T \times \mathbb{R}^n \to \mathbb{R}$ such that $F(t,\cdot)$ is strictly increasing for almost all $t \in T$. Then, for any $\mathcal{G} \in \mathbb{G}$, the set of F-Weighted Cournot-Nash equilibria is contained in the set of Pareto-Nash equilibria. This property and the previous results of stability for F-Weighted Cournot-Nash equilibria allow us to deduce stability properties for Pareto-Nash equilibria in $\mathbb{G}$.[7]

Theorem 5. *Let $\mathcal{T} = ((\mathbb{X},\tau),\kappa)$ be a parametrization of $\mathbb{G}$. Then, there is a dense residual set $\mathbb{X}' \subseteq \mathbb{X}$ such that, for all $X \in \mathbb{X}'$, $\kappa(X)$ has a $\mathcal{PN}$-essential equilibrium with respect to $\mathcal{T}$.*
Furthermore, for every parameter $X \in \mathbb{X}$, we have that:
(i) If $\Psi_{\mathcal{P}}(\kappa(X))$ is a singleton, then $\kappa(X)$ is $\mathcal{PN}$-essential with respect to $\mathcal{T}$.
(ii) $\Psi_{\mathcal{P}}(\kappa(X))$ has at least one minimal $\mathcal{PN}$-essential subset with respect to $\mathcal{T}$.
(iii) If $\Psi_{\mathcal{P}}(\kappa(X))$ has a connected $\mathcal{PN}$-essential set, then it has a $\mathcal{PN}$-essential component.

Proof. Let $F : T \times \mathbb{R}^n \to \mathbb{R}$ be the continuous function defined by $F(t,(y_1,\ldots,y_n)) = \sum_{k=1}^n y_k$. Since F is strictly increasing in y, for every $X \in \mathbb{X}$ the set of F-Weighted Cournot-Nash equilibria of $\kappa(X)$ is contained in its set of Pareto-Nash equilibria. Thus, if f^* is an F-essential equilibrium of $\kappa(X)$ with respect to $\mathcal{T}$, then it is a $\mathcal{PN}$-essential equilibrium of $\kappa(X)$ with respect to $\mathcal{T}$.

[7]These results extend Song and Wang (2010, Theorem 5) to large multi-objective generalized games.

It follows from Proposition 3 that there exists a dense residual set $\mathbb{X}' \subset \mathbb{X}$ such that, for every $X \in \mathbb{X}'$, all the F-Weighted Cournot-Nash equilibria of $\kappa(X)$ are F-essential with respect to $\mathcal{T}$. Thus, for each $X \in \mathbb{X}'$, $\kappa(X)$ has at least one $\mathcal{PN}$-essential equilibrium of $\kappa(X)$ with respect to $\mathcal{T}$.

Given $X \in \mathbb{X}$, $\Psi_F(\kappa(X)) \neq \emptyset$ and $\Psi_F(\kappa(X)) \subseteq \Psi_{\mathcal{P}}(\kappa(X))$. Hence, item (i) follows from Proposition 3(i). Moreover, as there exists $E \subseteq \Psi_F(\kappa(X))$ that is a minimal F-essential set with respect to $\mathcal{T}$ (Proposition 3(ii)), it follows that E is a $\mathcal{PN}$-essential subset of $\Psi_{\mathcal{P}}(\kappa(X))$ with respect to $\mathcal{T}$. Let $\mathcal{S}_E$ be the family of $\mathcal{PN}$-essential sets of $\Psi_{\mathcal{P}}(\kappa(X))$ with respect to $\mathcal{T}$ contained in E and ordered by set inclusion. Since $\mathcal{S}_E$ is non-empty and its elements are compact, any totally ordered subset of $\mathcal{S}_E$ has a lower bounded element. By Zorn's Lemma, $\mathcal{S}_E$ has a minimal element. This ensures that item (ii) holds. Finally, item (iii) follows from identical arguments to those made in Theorem 3(ii). □

7. Concluding Remarks

In this paper we address the theory of existence and essential stability of equilibria in large multi-objective generalized games. We prove the existence of Pareto-Nash and Weak Pareto-Nash equilibria under analogous assumptions to those required for Cournot-Nash equilibrium existence in large generalized games (Theorem 1). In addition, focusing on Weak Pareto-Nash equilibria, and imposing convexity assumptions on non-atomic players' characteristics, we obtain that for any continuous parametrization of the set of multi-objective games, essential components exist and multi-objective games are generically essential (Theorems 2, 3 and 4). Furthermore, previous results of the literature of essential stability for large generalized *games*, allow us to deduce some stability properties for the Pareto-Nash equilibria set (Theorem 5).

Appendix

Proof of Proposition 1. Given $\mathcal{G}((K_t, \Gamma_t, U_t)_{t\in T}) \in \mathbb{G}_{\mathcal{W}}$, let $B_t : \widehat{M} \twoheadrightarrow K_t$ be the Weak Pareto-Nash Best-reply correspondence for a player $t \in T$, i.e.,

$$B_t(m) := \left\{ f(t) \in \Gamma_t(m) : U_t(x(t), m) - U_t(f(t), m) \notin \mathbb{R}^n_{++}, \ \forall x(t) \in \Gamma_t(m) \right\}.$$

We will ensure that best-reply correspondences $(B_t)_{t\in T}$ have closed graph with non-empty and convex values. With this property, we can adapt the arguments of Correa and Torres-Martínez (2014, Theorem 2, Claim C) to ensure that the Weak Pareto-Nash equilibria set of $\mathcal{G}$ coincides with the fixed points of a closed graph correspondence.

Step 1. The correspondences $(B_t)_{t\in T}$ have closed graph and non-empty, compact and convex values. Note that, for any $t \in T$, B_t is analogous to any of the best-reply correspondences considered in Lin, Yang and Yu (2005), replacing the interior of a generic positive cone by the interior of $\mathbb{R}^n_+$. Therefore, it follows from the proof of Theorem 3.1 in Lin, Yang and Yu (2005) that: (i) Assumption (A6) implies that $\{B_t\}_{t\in T}$ have non-empty values; (ii) Assumption (A1) implies that $(B_t)_{t\in T}$ have closed graph; and (iii) Assumptions (A2) and (A7) imply that $(B_t)_{t\in T}$ have convex values. □

Step 2. The correspondence $\Omega_{\mathcal{G}} : \widehat{M} \twoheadrightarrow \widehat{M}$ given by $\Omega_{\mathcal{G}}(m) = \int_{T_1} H(t, B_t(m))d\mu$ has closed graph with convex and nonempty values.

Fix $m \in \widehat{M}$ and let $Q_m : T_1 \times \widehat{K} \to \mathbb{R}$ be the function defined by

$$Q_m(t, y) = \max_{x \in \Gamma_t(m)} \min_{1 \le i \le n} \{u_{t,i}(x, m) - u_{t,i}(y, m)\}.$$

It follows that, $y \in B_t(m)$ if and only if both $y \in \Gamma_t(m)$ and $Q_m(t, y) \le 0$. Under Assumption (A1), Berge's Maximum Theorem ensures that, for any $t \in T_1$, $Q_m(t, \cdot)$ is continuous. From Castaing and Valadier (1977, Lemma III.39, page 84) we can guarantee that, for any $y \in \widehat{K}$ the function $Q_m(\cdot, y)$ is $\mathcal{B}(T_1)$-measurable. Therefore, Q_m is $\mathcal{B}(T_1) \times \mathcal{B}(\widehat{K})$-measurable.

We conclude that $\{(t, y) \in T_1 \times \widehat{K} : y \in B_t(m)\} = \{(t, y) \in T_1 \times \widehat{K} : y \in \Gamma_t(m)\} \bigcap Q_m^{-1}((-\infty, 0))$ is measurable, which is a consequence of Assumption (A3). Hence, it follows from the Aumann's Selection Theorem (see Aliprantis and Border (2006, Theorem 18.26, page 608)) that there is a measurable function $f \in \widehat{\mathcal{F}}$ such that, for any $t \in T_1$, $f(t) \in B_t(m)$. Since H is continuous with respect to the product topology induced by the metrics of T_1 and $\widehat{K}$, it follows that $t \to H(t, f(t))$ is bounded and measurable. Therefore, $\Omega_{\mathcal{G}}$ has non-empty values.

On the other hand, as for any $m \in \widehat{M}$, $\Omega_{\mathcal{G}}(m)$ is the integral of $t \twoheadrightarrow H(t, B_t(m))$, it has convex values (see Aumann (1965)).

Finally, the closed graph property of $B_t(m)$ implies that $m \twoheadrightarrow H(t, B_t(m))$ has closed graph too. Then, since T_1 and $\widehat{K}$ are compact sets, and the function H is continuous, $m \twoheadrightarrow H(t, B_t(m))$ is also bounded. Therefore, it follows from Aumann (1976) that $\Omega_{\mathcal{G}}$ has closed graph. □

Define $\Phi_{\mathcal{G}} : \widehat{M} \twoheadrightarrow \widehat{M}$ by $\Phi_{\mathcal{G}}(m) = (\Omega_{\mathcal{G}}(m), (B_t(m))_{t \in T_2})$. Then, it follows from the previous arguments that $\Phi_{\mathcal{G}}$ has closed graph with non-empty, convex and compact values. Applying Kakutani-Fan-Glicksberg Fixed Point Theorem, we conclude that the set of fixed points of $\Phi_{\mathcal{G}}$ is non-empty and compact. Note that, $m^* \in \Phi_{\mathcal{G}}(m^*)$ if and only if there exists $f^*(t) \in \widehat{\mathcal{F}}$ such that $m^* = \left(\int_{T_1} H(t, f^*(t))d\mu, (f^*(t))_{t \in T_2}\right)$, with $f^*(t) \in B_t(m^*)$ for almost all $t \in T$. Therefore, the fixed points of $\Phi_{\mathcal{G}}$ are the Weak Pareto-Nash equilibrium messages of $\mathcal{G}$, $\Psi_{\mathcal{W}}(\mathcal{G})$. *Q.E.D.*

Proof of Proposition 2. Since $\widehat{M}$ is compact, to ensure that $\Psi_{\mathcal{W}} : \mathbb{G}_{\mathcal{W}} \twoheadrightarrow \widehat{M}$ is upper hemicontinuous we only need to guarantee that $\text{Graph}(\Psi_{\mathcal{W}}) := \left\{(\mathcal{G}, m) \in \mathbb{G}_{\mathcal{W}} \times \widehat{M} \ : \ m \in \Phi_{\mathcal{G}}(m)\right\}$ is a closed set. Let $\{(\mathcal{G}_r, m_r)\}_{r \in \mathbb{N}} \subset \text{Graph}(\Psi_{\mathcal{W}})$ be a sequence that converges to $(\overline{\mathcal{G}}, \overline{m}) \in \mathbb{G}_{\mathcal{W}} \times \widehat{M}$, where $\overline{\mathcal{G}} = ((\overline{K}_t, \overline{\Gamma}_t, \overline{U}_t)_{t \in T})$ and, for any $r \in \mathbb{N}$, $\mathcal{G}_r = ((K_t^r, \Gamma_t^r, U_t^r)_{t \in T})$. We want to prove that $(\overline{\mathcal{G}}, \overline{m}) \in \text{Graph}(\Psi_{\mathcal{W}})$.

It follows from Proposition 1 that, for any $r \in \mathbb{N}$, $m_r \in \Phi_{\mathcal{G}_r}(m_r)$. Thus, there exists a strategy profile $f_r \in \mathcal{F}$ such that $m_r = \left(\int_{T_1} H(t, f_r(t))d\mu, (f_r(t))_{t \in T_2}\right)$, with $H(\cdot, f_r(\cdot))$ measurable. In addition, there is a full measure set $T_r \subseteq T$ such that, for every $t \in T_r$ we have that $f_r(t) \in B_t^r(m_r)$, where B_t^r is the Weak Pareto-Nash Best-reply correspondence of player t in $\mathcal{G}_r$ (see the proof of Proposition 1).

Note that, for every $r \in \mathbb{N}$, $(B_t^r)_{t \in T}$ have non-empty values. Therefore, for any $t \in T \setminus T_r$ we can replace $f_r(t)$ by an action in $B_t^r(m_r)$, without affecting the properties of m_r. For these reasons, and without loss of generality, we can assume that, for every $r \in \mathbb{N}$ and for every $t \in T$, $f_r(t) \in B_t^r(m_r)$.

By the same arguments of Correa and Torres-Martínez (2014, Theorem 1, Step 1), the multi-dimensional Fatou's Lemma ensures that there is $\overline{f} \in \mathcal{F}$ such that $\overline{m} = \left(\int_{T_1} H(t,\overline{f})d\mu, (\overline{f}(t))_{t\in T_2}\right)$ and, for all players in a full measure set $\widetilde{T} \in T$ the following properties hold: (i) $\overline{f}(t) \in \overline{\Gamma}_t(\overline{m})$; and (ii) there is a subsequence $\{f_r(t)\}_{r\in\mathbb{N}_t}$ that converges to $\overline{f}(t)$.

Note that, $(\overline{\mathcal{G}}, \overline{m}) \in \text{Graph}(\Psi_{\mathcal{W}}) \iff \overline{f}(t) \in B_t(\overline{m})$, for almost all $t \in T$.

By contradiction, assume that there exists a positive measure set $\widetilde{T}^* \subseteq \widetilde{T}$ such that, $\overline{f}(t) \notin B_t(\overline{m})$, $\forall t \in \widetilde{T}^*$. Then, given $t \in \widetilde{T}^*$, there exists $\overline{x}(t) \in \overline{\Gamma}_t(\overline{m})$ and $\delta_t > 0$ such that, $\overline{U}_t(\overline{x}(t),\overline{m}) \gg \overline{U}_t(\overline{f}(t),\overline{m})+\delta_t(1,\ldots,1)$. This implies that, for an $r \in \mathbb{N}_t$ large enough, $\overline{U}_t(\overline{x}(t),m_r) \gg \overline{U}_t(f_r(t),m_r)+\delta_t(1,\ldots,1)$. Furthermore, there exists $r_0 \in \mathbb{N}$ such that, for all $r \in \mathbb{N}_t$ with $r > r_0$ we have that $U_t^r(\overline{x}(t),m_r) \gg U_t^r(f_r(t),m_r) + \frac{\delta_t}{2}(1,\ldots,1)$.[8] In addition, it is always possible to find a sequence $\{x_r(t)\}_{r\in\mathbb{N}_t} \subset \widehat{K}$ that converges to $\overline{x}(t)$ and satisfies $x_r(t) \in \Gamma_t^r(m_r)$, for any $r \in \mathbb{N}_t$.[9]

Therefore, it follows from inequalities above that,

$$(U_t^r(\overline{x}(t),m_r) - U_t^r(x_r(t),m_r)) + (U_t^r(x_r(t),m_r) - U_t^r(f_r(t),m_r)) \gg \frac{\delta_t}{2}(1,...,1).$$

The first part of the right side of the inequality above tends to zero as $r \in \mathbb{N}_t$ goes to infinity.[10] Hence, there exists $\overline{r}_t > 0$ such that, for any $r \in \mathbb{N}_t$ with $r > \overline{r}_t$, $f_r(t) \notin B_t^r(m_r)$, which is a contradiction.

Thus, we ensure that $\text{Graph}(\Psi_{\mathcal{W}})$ is closed, which at the same time implies that $\Psi_{\mathcal{W}}$ is upper hemi-continuous. Since $(\mathbb{G}_{\mathcal{W}},\rho)$ is a complete metric space, it follows from Lemma 6 in Carbonell-Nicolau (2010) that there exists a dense residual subset of $\mathbb{G}_{\mathcal{W}}$ in which $\Psi_{\mathcal{W}}$ is continuous. *Q.E.D.*

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[8] Indeed, for $r \in \mathbb{N}_t$ large enough,

$$\delta_t(1,...,1) \ll \overline{U}_t(\overline{x}(t),m_r) - U_t^r(\overline{x}(t),m_r) + U_t^r(\overline{x}(t),m_r) - U_t^r(f_r(t),m_r) + U_t^r(f_r(t),m_r) - \overline{U}_t(f_r(t),m_r),$$

which implies that $\delta_t(1,...,1) \ll U_t^r(\overline{x}(t),m_r) - U_t^r(f_r(t),m_r) + 2d(\overline{U}_t,U_t^r)(1,\ldots,1)$. Since U_t^r converges to $\overline{U}_t$, we conclude that, for $r \in \mathbb{N}_t$ large enough, $U_t^r(\overline{x}(t),m_r) \gg U_t^r(f_r(t),m_r) + \frac{\delta_t}{2}(1,\ldots,1)$.

[9] The lower hemi-continuity of $\overline{\Gamma}_t$ ensures that, for any $r \in \mathbb{N}_t$ there is $\tilde{x}_r(t) \in \overline{\Gamma}_t(m_r)$ such that $\tilde{x}_r(t) \to_r \overline{x}(t)$. Fix $r \in \mathbb{N}_t$. Since $\Gamma_t^s(m_r) \to_s \overline{\Gamma}_t(m_r)$, there exists a sequence $\{y_n^s(t)\}_{s\in\mathbb{N}_t}$ such that, for each $s \in \mathbb{N}_t$, $y_r^s(t) \in \Gamma_t^s(m_r)$ and $y_r^s(t)$ converges to $\tilde{x}_r(t)$ as s goes to infinity. We affirm that $x_s(t) := y_s^s(t) \to_s \overline{x}(t)$. Indeed, we have that $d(x_s(t),\overline{x}(t)) \leq d(y_s^s(t),y_s^v(t)) + d(y_s^v(t),\tilde{x}_s(t)) + d(\tilde{x}_s(t),\overline{x}(t))$. If we fix s, for v large enough, the second term on the right side of the inequality becomes increasingly smaller. Moreover, if we have a sufficiently large s, the third term becomes smaller. Therefore, it is sufficient to prove that for large enough and uncorrelated s and v, the first term on the right side converges to zero. Notice that $d(y_s^s,y_s^v) \leq d(\Gamma_t^s(m_s),\Gamma_t^v(m_s)) \leq d(\Gamma_t^s,\Gamma_t^v)$. Since the sequence $\{\Gamma_t^r\}_{r\in\mathbb{N}}$ converges, we obtain the result.

[10] Given $r \in \mathbb{N}_t$, let $A_r = U_t^r(\overline{x}(t),m_r) - U_t^r(x_r(t),m_r)$. Then,

$$A_r \quad = \quad U_t^r(\overline{x}(t),m_r) - \overline{U}_t(\overline{x}(t),m_r) + \overline{U}_t(\overline{x}(t),m_r) - \overline{U}_t(x_r(t),m_r) + \overline{U}_t(x_r(t),m_r) - U_t^r(x_r(t),m_r),$$

which implies that $\|A_r\| \leq 2d(U_t^r,\overline{U}_t)+\|\overline{U}_t(\overline{x}(t),m_r)-\overline{U}_t(x_r(t),m_r)\|$. Taking the limit when $r \in \mathbb{N}_t$ goes to infinity, we guarantee that A_r converges to zero.

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