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ON HOUSING MARKETS WITH INDECISIVE AGENTS

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ON HOUSING MARKETS WITH INDECISIVE AGENTS

EMILIO GUAMÁN AND JUAN PABLO TORRES-MARTÍNEZ

Abstract. We study the non-monetary exchange of indivisible goods when agents may be unable to compare some of them. Adding incomplete preferences to the Shapley-Scarf housing market model, we introduce two concepts of coalitional stability: the *core* and the *strong core*. The *core* is the set of allocations immune to blocking coalitions that improve the well-being of house-switching members, while the *strong core* is the set of allocations immune to blocking coalitions that may leave some members with a house incomparable with the original. In the domain of incomplete, transitive, and strict preferences, we characterize a family of group strategy-proof mechanisms that always select allocations in the core. Moreover, in the subdomain in which incomplete preferences induce transitive incomparability relations, we show that there are efficient, individually rational, and weakly group strategy-proof mechanisms that select allocations in the strong core when it is non-empty. We also extend these results to housing allocation problems in which existing tenants and newcomers coexist.

Keywords: Housing markets; Incomplete preferences; Mechanism design.
JEL Classification: D47, C78.

1. Introduction

There are many situations in which the allocation of indivisible goods is done without the use of money. For instance, the online bartering of objects, the allocation of on-campus housing to students, the matching of patients with donors for kidney transplants, or the reallocation of students between partner universities in exchange programs. Matching theory provides the necessary toolbox to study these economic problems. Coalitional stability and incentives have been studied in these scenarios based on the models introduced by Shapley and Scarf (1974), Hylland and Zeckhauser (1979), Abdulkadiroğlu and Sönmez (1999), and Roth, Sönmez, and Ünver (2004).

The focus of this literature has always been on situations in which agents have complete preferences.[1] However, given the lack of information, the complexity of obtaining it, or the absence of incentives to be informed, there are scenarios in which some individuals cannot compare all the available alternatives.

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[1]Previous studies have incorporated incomplete preferences in *school choice problems* and *marriage markets*. Kitahara and Okumura (2021, 2023a, 2023b) study the effects of the presence of schools with incomplete priorities on stability and efficiency. In many-to-one matching markets with contracts, Che, Kim, and Kojima (2021) study stability and monotone comparative statics allowing for incomplete preferences. Recently, Kuvalekar (2022) studies the effects of incompleteness on the existence of coalitionaly stable outcomes in marriage markets.

Moreover, incomplete preferences arise naturally in multi-objective decision making (Ok (2002)), where agents evaluate alternatives considering various attributes that may not be easily aggregated.

It is important to remark that a model with incomplete preferences cannot be reduced to a scenario with weak preferences. As pointed out by Aumann (1962), "indifference between two alternatives should not be confused with incomparability; the former involves a positive decision that it is immaterial whether the one or the other alternative is chosen, whereas the latter means that no decision is reached" (cf., Mandler (2005), Eliaz and Ok (2006), Arlegi, Bourgeois-Gironde, and Hualde (2022)). In addition, the incomparability relations induced by an incomplete preference are not necessarily transitive.[2] Hence, even from a mathematical point of view, they cannot be identified with the indifference relations induced by a complete and transitive preference.[3]

In this paper, we include agents with incomplete preferences in Shapley and Scarf (1974) housing markets, a framework where the non-monetary exchange of indivisible goods is studied through a parsimonious model in which every agent owns a "house" that can be exchanged for the property of any of the other individuals. Shapley and Scarf (1974) focus on the existence of coalitionally stable allocations: distributions of houses among agents such that no group can deviate to improve the well-being of *all* of its members. Assuming that agents have complete, transitive, and weak preferences for houses, they show that the set of coalitionally stable allocations is non-empty and that David Gale's *Top Trading Cycles Algorithm* (TTC) can be used to obtain one of them. A natural refinement of the solution concept proposed by Shapley and Scarf is the *strict core*, defined as the set of housing allocations such that no coalition can deviate to improve the well-being of one of its members without worsening the others. When preferences for houses are complete, transitive, and strict, Roth and Postlewaite (1977) show that the strict core is a singleton that can be found by applying the TTC algorithm.[4] Moreover, TTC determines the only mechanism that is Pareto efficient, individually rational, and strategy-proof in this context (see Ma (1994)).

Some questions arise in the presence of indecisive agents: What are the natural extensions of the strict core concept? To what extent the blocking power of coalitions is affected by the participation of agents that cannot compare their situation before and after a deviation? Are there mechanisms that do not incentivize agents to misreport their preferences by announcing different degrees of incompleteness?

We propose two extensions of the strict core to Shapley-Scarf housing markets with incomplete, transitive, and strict preferences: the *core* and the *strong core*. The core is the set of allocations that are immune to blocking coalitions that improve the well-being of every house-switching member. The strong core is the set of allocations that are immune to blocking coalitions that may leave some members with a house incomparable with the original.

To define the core it is assumed that members of a blocking coalition can compare the houses they receive before and after a deviation. Essentially, when preferences are incomplete, it may be reasonable to presume that the only agents that participate in a deviation are those who know how to contrast the alternatives involved. As a consequence, the core has no counterpart in housing markets with complete, transitive, and weak preferences (cf., Alcalde-Unzu and Molis (2011), Jaramillo and Manjunath (2012)),

[2]To illustrate this point, suppose that an agent evaluates the alternatives in $A = \{a_1, a_2, a_3\}$ considering two attributes that she cannot aggregate into a single preference. That is, there are functions $f, g : A \to \mathbb{R}$ such that a is preferred to a' if and only if $(f(a), g(a)) > (f(a'), g(a'))$. Hence, when $f(a_2) > f(a_3) > f(a_1)$ and $g(a_3) > g(a_1) > g(a_2)$, the alternative a_2 is incomparable with both a_1 and a_3. However, these incomparability relations are intransitive, since a_3 is preferred to a_1.

[3]When intransitive preferences are allowed, *incomparability* and *indifference* are indistinguishable from a mathematical perspective. However, even in this scenario, an indecisive agent may behave differently than an indifferent one (e.g., when evaluating whether to participate in a blocking coalition).

[4]Roth and Postlewaite (1977) refers to the strict core as the *core defined by weak domination*.

because in the latter context it is not credible to force coalitions to leave out those who are indifferent between the alternatives involved.[5]

On the other hand, the strong core is based on the idea that any agent who does not know how to compare two houses will be available to swap them. At a first glance, when preference incompleteness is high, this behavior may significantly reduce the set of coalitionally stable allocations.

Our concepts of coalitional stability are single-valued and coincide with the strict core when agents have complete, transitive, and strict preferences. However, under incomplete preferences, the core may have more than one element and the strong core may be empty (see Examples 1 and 2).

By adapting to our framework the techniques developed by Kuvalekar (2022) in marriage markets with incomplete preferences, we show that the core can be algorithmically constructed and coincides with the collection of allocations obtained by applying the TTC algorithm to the set of *completions* of agents' preferences (Proposition 1).[6] This result allows us to show that the core is a singleton only if the strong core is non-empty and coincides with it. Moreover, the core weakly increases with the incompleteness of preferences, a property that the strong core does not satisfy (see Corollary 1 and Example 4). As in Shapley-Scarf housing markets with complete preferences (cf., Roth and Postlewaite (1977)), every allocation in the (strong) core is *ex-post stable* (see Remark 2).

From the point of view of mechanism design, we show that there are many group strategy-proof mechanisms that select allocations in the core. Indeed, a mechanism with these properties can be obtained by using personalized protocols to transform agents' incomplete preferences into strict linear orders and then applying the TTC algorithm to the preference profile obtained (Theorem 1).[7] This result follows from the group strategy-proofness of TTC in the domain of complete, transitive, and strict preferences (see Bird (1994) and Moulin (1995)). We also show that no mechanism that select allocations in the core prevents the existence of a group of agents who want to misreport their preferences to improve the well-being of at least one of them without worsening the situation of the others—in the sense that some of them may leave with a house incomparable with the original (see Remark 3).

Notice that, when the relations of incomparability induced by incomplete preferences are transitive, the strong core coincides with the strict core of the Shapley-Scarf housing market with complete preferences in which incomparability is identified with indifference. Therefore, the subdomain of preferences in which the strong core is non-empty can be identified with a superset of the collection of preference profiles in which the strict core is non-empty (cf., Quint and Wako (2004)). Moreover, for the preference profiles for which incomparability can be identified with indifference, the results of Wako (1991) for the strict core allow us to show that the strong core is essentially single-valued (see Remark 4).

Assuming that agents have complete, transitive, and weak preferences, Alcalde-Unzu and Molis (2011), Jaramillo and Manjunath (2012), and Plaxton (2013) introduce Pareto efficient, individually rational, and strategy-proof mechanisms that select an allocation in the strict core when it is non-empty. Among the algorithms studied by these authors, the *Top Trading Absorbing Sets Mechanisms* (TTAS) and the *Top Cycles Mechanisms* (TC) are weakly group strategy-proof (see Ahmad (2021)).

We adapt these results to guarantee that there are efficient, individually rational, and weakly group strategy-proof mechanisms that select allocations in the strong core when it is non-empty. More precisely,

[5]Furthermore, in the absence of transaction costs, there is no reason for an agent to refrain from participating in a blocking coalition that secures her a house that she considers indifferent to the original (see Remark 1).

[6]A *completion* is a profile of strict linear orders that respect agents' preferences. Following the ideas of Szpilrajn (1930), we introduce the *Sequential Completion Algorithm* to compute all the completions of a preference profile.

[7]When agents have complete, transitive, and weak preferences, a rankings of houses can be used for tie-breaking indifferences (Elhers (2014)). In our framework, we cannot apply an analogous strategy to eliminate incomparability without compromising the transitivity of the completion obtained (see the remarks after Theorem 1).

in the domain of preferences in which the relations of incomparability are transitive, mechanisms with these properties are obtained when the TTAS and the TC algorithms are applied to the complete preferences generated by identifying incomparability with indifference (Theorem 2).

Our analysis can be extended to study efficiency and incentives in the housing allocation problems introduced by Hylland and Zeckhauser (1979) and Abdulkadiroğlu and Sönmez (1999). By appealing to the results of Svensson (1994, 1999) and Pápai (2000) for housing allocation problems with complete preferences, we characterize a family of (group) strategy-proof mechanisms that implement (weakly) efficient allocations when agents have incomplete preferences (see Section 6).[8]

More precisely, in housing allocation problems in which existing tenants and newcomers coexists, a weakly efficient, individually rational, and group strategy-proof mechanism can be obtained by applying the *You Request My House–I Get Your Turn* mechanism introduced by Abdulkadiroğlu and Sönmez (1999) to some completion of agents' preferences (cf., Sönmez and Ünver (2005, 2010)). Moreover, in the domain of preferences in which incomparability relations are transitive, an efficient, individually rational, and weakly group strategy-proof mechanism can be obtained by applying any of the TC algorithms introduced by Jaramillo and Manjunath (2012) to the profiles of complete preferences obtained by identifying incomparability with indifference. For the housing allocation problems of Hylland and Zeckhauser (1979), analogous mechanisms can be constructed based on the application of a *Serial Dictatorship Rule* to a completion of preferences or to a profile of complete preferences obtained by identifying incomparability with indifference (when it is possible to do so).

The rest of the paper is organized as follows. In Sections 2 and 3 the core and the strong core are introduced and characterized. In Section 4 we discuss some properties of efficiency and ex-post stability. In Section 5 we study the incentive properties of mechanisms that implement allocations in either the core or the strong core. Our analysis is extended to housing allocation problems in Section 6, and we include comments on topics for future research in Section 7. Some proofs are in the Appendix.

2. Model

Consider a *Shapley-Scarf housing market with incomplete preferences* $[I, H, (\succ_i)_{i\in I}]$ in which there is a set $I = \{1, \dots, n\}$ of agents and a collection $H = \{h_1, \dots, h_n\}$ of houses. The house h_i is owned by agent i and the set $\succ_i \subseteq H \times H$ are the pairs of houses that she can compare, in the sense that (h, h') belongs to $\succ_i$ whenever i strictly prefers h to h'. Throughout the text, $h \succ_i h'$ indicates that $(h, h') \in \succ_i$. The preferences induced by $\succ_i$ are transitive: if $h \succ_i h'$ and $h' \succ_i h''$, then $h \succ_i h''$.

An agent i considers h and h' *incomparable*—denoted by $h \otimes_i h'$—when neither $h \succ_i h'$ nor $h' \succ_i h$. The relation $\otimes_i$ is not necessarily transitive: there may exist $h, h', h'' \in H$ such that $h \otimes_i h'$, $h' \otimes_i h''$, and $h \succ_i h''$. For this reason, by equating *incomparability* with *indifference*, we cannot always identify the preference relation induced by $\succ_i$ with a complete, transitive, and weak preference relation.

Let $M(\succ_i) = \{(h_j, h_k) \in H\times H : h_j \otimes_i h_k,\ j < k\}$ be the pairs of houses that i considers incomparable. An agent i has *incomplete preferences* as long as $M(\succ_i)$ is non-empty.

A *housing allocation* is characterized by a function $\mu : I \to H$ that associates a different house to each agent. Let $\mathcal{A}$ be the set of housing allocations. A coalition is a non-empty subset of I. Given a coalition C, let $e(C) = \{h_k \in H : k \in C\}$ be the set of houses that are property of the agents in C. We refer to any bijective function $\sigma : C \to e(C)$ as an *agreement* among the members of C. Occasionally, an agreement σ will be identified with the family of pairs $(i, \sigma(i))$, with $i \in C$.

[8] In our framework, there are two natural concepts of efficiency depending on whether a redistribution of houses may leave some agents with an alternative incomparable with the original: *weak efficiency* and *efficiency* (see Section 4). These concepts are equivalent to Pareto efficiency when preferences are complete.

In this context, consider the following concepts of coalitional stability:

- A housing allocation μ is *blocked* by a coalition C if there is an agreement $\sigma : C \to e(C)$ among the members of C such that:
  (i) For all $i \in C$ we have that $\sigma(i) \succ_i \mu(i)$ or $\sigma(i) = \mu(i)$.
  (ii) There exists $i \in C$ such that $\sigma(i) \succ_i \mu(i)$.

  The *core* $\mathcal{C}(\succ)$ is the set of allocations in $\mathcal{A}$ that are not blocked by any coalition.
- A housing allocation μ is *weakly blocked* by a coalition C if there is an agreement $\sigma : C \to e(C)$ among the members of C such that:
  (i) For all $i \in C$ we have that $\sigma(i) \succ_i \mu(i)$, $\sigma(i) = \mu(i)$, or $\sigma(i) \otimes_i \mu(i)$.
  (ii) There exists $i \in C$ such that $\sigma(i) \succ_i \mu(i)$.

  The *strong core* $\mathcal{C}_{\mathrm{S}}(\succ)$ is the set of allocations in $\mathcal{A}$ that are not weakly blocked by any coalition.

Evidently, the strong core is a refinement of the core. Moreover, when $\succ$ is a profile of complete, transitive, and strict preferences, $\mathcal{C}(\succ)$ and $\mathcal{C}_{\mathrm{S}}(\succ)$ coincide with the *strict core*, that we denote by $\mathbb{K}(\succ)$.

The definition of $\mathcal{C}(\succ)$ implicitly restricts the blocking coalitions: if $C \subseteq I$ blocks an allocation μ through an agreement σ, then each member of C needs to be able to compare the houses that she receives under μ and σ. Essentially, when preferences are incomplete, it is reasonable to assume that the only agents that participate in a deviation are those who know how to compare the alternatives involved.

In the particular case in which $(\otimes)_{i\in I}$ are transitive relations, $\mathcal{C}(\succ)$ is a subset of the *core* and a superset of the *strict core* of the Shapley-Scarf housing market in which incomparability is identified with indifference. Moreover, the strict core of this housing market coincides with $\mathcal{C}_{\mathrm{S}}(\succ)$, a relationship that will allow us to analyze incentive properties of mechanisms that select allocations in the strong core when it is non-empty (see Section 5).

The following example illustrates our concepts of coalitional stability.

Example 1. Consider a market $[I, H, (\succ_i)_{i\in I}]$ with four agents and preferences characterized by

$$\begin{aligned}
\succ_1&: \quad h_2 \succ_1 h_3 \succ_1 h_1, \quad h_4 \succ_1 h_3 \succ_1 h_1, \quad h_2 \otimes_1 h_4;\\
\succ_2&: \quad h_1 \succ_2 h_2 \succ_2 h_3 \succ_2 h_4;\\
\succ_3&: \quad h_1 \succ_3 h_4 \succ_3 h_3 \succ_3 h_2;\\
\succ_4&: \quad h_2 \succ_4 h_4 \succ_4 h_1, \quad h_2 \succ_4 h_3 \succ_4 h_1, \quad h_3 \otimes_4 h_4.
\end{aligned}$$

Given $\mu \in \mathcal{A}$ we have that:

- If $\mu(2) \in \{h_3, h_4\}$, then it is blocked by the coalition $\{2\}$ through the agreement $[(2, h_2)]$.
- If $\mu(3) = h_2$, then it is blocked by the coalition $\{3\}$ through the agreement $[(3, h_3)]$.
- If $\mu(4) = h_1$, then it is blocked by the coalition $\{4\}$ through the agreement $[(4, h_4)]$.

Since $\{h_2, h_4\}$ are the top choices for agent 1 and h_1 is the best alternative for agent 2, it follows that:

- The housing allocations $[(1, h_1), (2, h_2), (3, h_3), (4, h_4)]$, $[(1, h_1), (2, h_2), (3, h_4), (4, h_3)]$, and $[(1, h_3), (2, h_1), (3, h_4), (4, h_2)]$ are blocked by $\{1, 2\}$ through the agreement $[(1, h_2), (2, h_1)]$.
- The housing allocations $[(1, h_3), (2, h_2), (3, h_1), (4, h_4)]$ and $[(1, h_4), (2, h_2), (3, h_1), (4, h_3)]$ are blocked by the coalition $\{1, 2, 4\}$ through the agreement $[(1, h_4), (2, h_1), (4, h_2)]$.

We conclude that $\mathcal{C}_{\mathrm{S}}(\succ) \subseteq \mathcal{C}(\succ) \subseteq \{\mu_1, \mu_2, \mu_3\}$, where

$$\mu_1 = [(1, h_2), (2, h_1), (3, h_3), (4, h_4)], \qquad \mu_2 = [(1, h_2), (2, h_1), (3, h_4), (4, h_3)],$$

$$\mu_3 = [(1, h_4), (2, h_1), (3, h_3), (4, h_2)].$$

Although the housing allocations μ_1 and μ_2 cannot be blocked by any coalition, μ_1 is weakly blocked by the coalition $\{3,4\}$ through the agreement $[(3,h_4),(4,h_3)]$, and μ_2 is weakly blocked by the coalition $\{1,2,4\}$ through the agreement $[(1,h_4),(2,h_1),(4,h_2)]$. Moreover, it is not difficult to verify that μ_3 cannot be weakly blocked by any coalition. Therefore, $\mathcal{C}(\succ)=\{\mu_1,\mu_2,\mu_3\}$ and $\mathcal{C}_{\mathrm{S}}(\succ)=\{\mu_3\}$. □

Unlike the case with complete and strict preferences, it follows from Example 1 that the core $\mathcal{C}(\succ)$ is not necessarily a singleton when preferences are incomplete. Moreover, although the *strong core* is non-empty in this example, only $\mathcal{C}(\succ)$ will be always non-empty. Indeed, the next example describes a market where no housing allocation belongs to the strong core.

Example 2. Let $[I,H,(\succ_i)_{i\in I}]$ be a housing market with three agents and preferences characterized by

$$\succ_1\colon\ h_3\succ_1 h_1\succ_1 h_2;\qquad \succ_2\colon\ h_3\succ_2 h_2\succ_2 h_1;\qquad \succ_3\colon\ h_1\succ_3 h_3,\ \ h_2\succ_3 h_3,\ \ h_1\otimes_3 h_2.$$

Since h_3 is the best alternative for agents 1 and 2, for any $\mu\in\mathcal{A}$ there exists $i\in\{1,2\}$ such that $h_3\succ_i\mu(i)$. Hence, as agent 3 cannot compare h_1 and h_2, the coalition $C=\{i,3\}$ weakly blocks μ through the agreement that interchanges the properties of i and 3. This implies that $\mathcal{C}_{\mathrm{S}}(\succ)=\emptyset$. □

A preference profile $(\widehat{\succ}_i)_{i\in I}$ is a *completion* of $(\succ_i)_{i\in I}$ when the following conditions hold:

- For each agent i, $\widehat{\succ}_i$ is a complete, transitive, and strict preference defined on H.
- For each agent i, $h\,\widehat{\succ}_i\,h'$ whenever $h\succ_i h'$.

Let $\mathrm{Co}(\succ)$ be the non-empty set of completions of $\succ=(\succ_i)_{i\in I}$.[9]

Notice that, if a coalition C blocks a housing allocation μ when preferences are $\succ$, then C also blocks μ under every $\widehat{\succ}\in\mathrm{Co}(\succ)$. Also, if C weakly blocks μ when preferences are $\succ$, then there exists $\widehat{\succ}\in\mathrm{Co}(\succ)$ such that C blocks μ under $\widehat{\succ}$. Therefore, we have that

$$\bigcup_{\widehat{\succ}\in\mathrm{Co}(\succ)}\mathbb{K}(\widehat{\succ})\subseteq\mathcal{C}(\succ),\qquad\qquad \bigcap_{\widehat{\succ}\in\mathrm{Co}(\succ)}\mathbb{K}(\widehat{\succ})\subseteq\mathcal{C}_{\mathrm{S}}(\succ),$$

where $\mathbb{K}(\widehat{\succ})$ is the strict core of the Shapley-Scarf housing market in which preferences are $\widehat{\succ}$.

It is well-known that $\mathbb{K}(\widehat{\succ})$ is a singleton that can be obtained by the application of David Gale's *Top Trading Cycles algorithm* (TTC) to the preference profile $\widehat{\succ}$ (see Roth and Postlewaite (1977)). Hence, the core $\mathcal{C}(\succ)$ is always non-empty.

3. Characterization of $\mathcal{C}(\succ)$ and $\mathcal{C}_{\mathrm{S}}(\succ)$

In this section we characterize the core $\mathcal{C}(\succ)$ and the strong core $\mathcal{C}_{\mathrm{S}}(\succ)$. To achieve this goal, we introduce the *Sequential Completion Algorithm* (SC), which essentially describes the process applied by Szpilrajn (1930) to complete a preference without compromising transitivity.[10] The association of incomplete preferences to their completions will be fundamental in our results.

[9] The *extension lemma* of Szpilrajn (1930) shows that any profile of incomplete, transitive, and strict preferences has a completion (cf., Fishburn (1970, Theorem 2.4)).

[10] Kitahara and Okumura (2023) and Okumura (2023) propose an alternative algorithm to generate the set of completions of a preference profile.

Sequential Completion Algorithm (SC)

Given a preference profile $(\succ_i)_{i\in I}$, apply the following procedure for each $i \in I$:

- **Step 1:** For every $a, b \in H$ such that $a \succ_i b$, define $a \,\widehat{\succ}_i\, b$. Let $Z(\succ_i) = M(\succ_i)$.
- **Step 2:** Fix $(a, b) \in Z(\succ_i)$ and define either $a \,\widehat{\succ}_i\, b$ or $b \,\widehat{\succ}_i a$.
  Step 2.1: If $a \,\widehat{\succ}_i\, b$ was defined, apply the following rules:
  (1) If there exists $c \in H$ such that $c \succ_i a$ and $c \otimes_i b$, define $c \,\widehat{\succ}_i\, b$.
  (2) If there exists $c \in H$ such that $b \succ_i c$ and $a \otimes_i c$, define $a \,\widehat{\succ}_i\, c$.
  Step 2.2: If $b \,\widehat{\succ}_i\, a$ was defined, apply the following rules:
  (1) If there exists $c \in H$ such that $c \succ_i b$ and $c \otimes_i a$, define $c \,\widehat{\succ}_i\, a$.
  (2) If there exists $c \in H$ such that $a \succ_i c$ and $b \otimes_i c$, define $b \,\widehat{\succ}_i\, c$.
- **Step 3:** Eliminate from $Z(\succ_i)$ the pairs $(h_j, h_k) \in H \times H$, with $j < k$, for which it was defined in the previous step that either $h_j \,\widehat{\succ}_i\, h_k$ or $h_k \,\widehat{\succ}_i\, h_j$.
- **Step 4:** Repeat Steps 2 and 3 until $Z(\succ_i) = \emptyset$.

By construction, the application of the SC algorithm to $(\succ_i)_{i\in I}$ produces a completion of it.

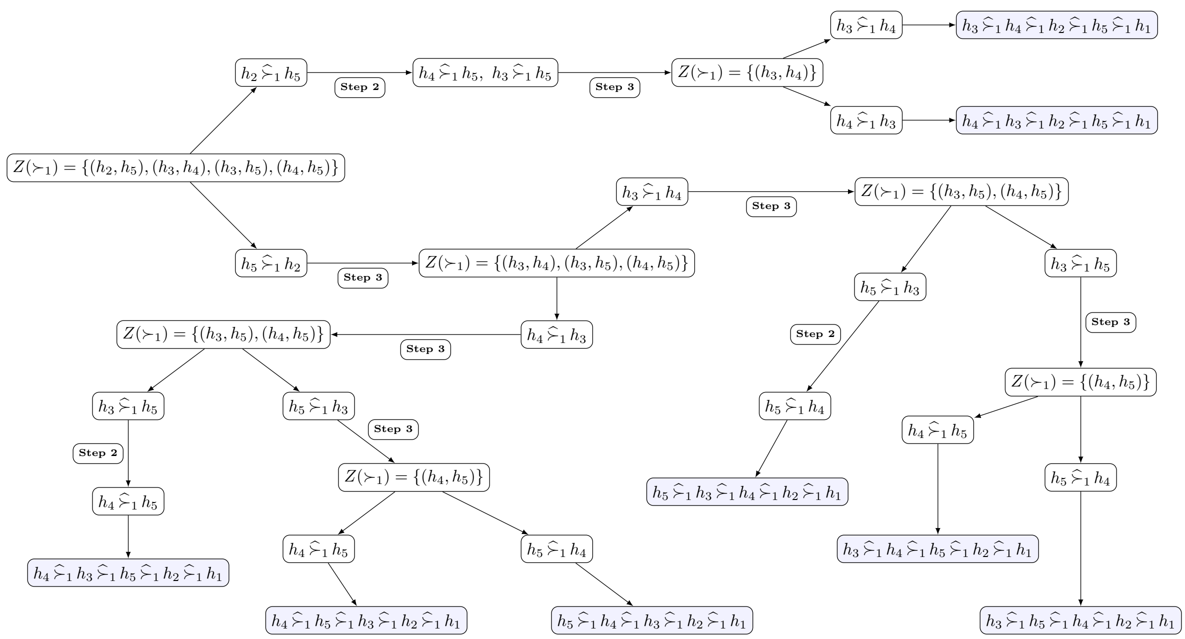
Since the decisions taken at Step 2 of the SC algorithm may affect the outcome, we denote by $\text{SC}(\succ)$ the set of preferences profiles that can be generated when the Sequential Completion algorithm is applied to a preference profile $\succ = (\succ_i)_{i\in I}$. In Proposition 1 we will show that $\text{Co}(\succ) = \text{SC}(\succ)$.

The next example illustrates the implementation of the SC algorithm.

Example 3. Consider a Shapley-Scarf housing market with five houses and where agent 1 has incomplete preferences characterized by

$$h_3 \succ_1 h_2 \succ_1 h_1, \quad h_4 \succ_1 h_2 \succ_1 h_1, \quad h_5 \succ_1 h_1, \;\; h_2 \otimes_1 h_5, \quad h_3 \otimes_1 h_4, \quad h_3 \otimes_1 h_5, \quad h_4 \otimes_1 h_5.$$

Since $M(\succ_1) = \{(h_2, h_5), (h_3, h_4), (h_3, h_5), (h_4, h_5)\}$, there are eight forms to completing $\succ_1$ without compromising transitivity. The following figure shows how the SC algorithm can be used to obtain these complete, transitive, and strict preferences:



The Sequential Completion Algorithm

The following result characterizes $\mathcal{C}(\succ)$ and $\mathcal{C}_{\mathrm{S}}(\succ)$ in terms of the *strict cores* of the Shapley-Scarf housing markets in which agents' preferences are given by completions of $\succ$. Its proof adapts to our framework the techniques applied by Kuvalekar (2002) to characterize coalitionally stable matchings in marriage markets with incomplete preferences (see Appendix A).

Proposition 1. *For every preference profile* $\succ = (\succ_i)_{i\in I}$, *we have that*

$$\mathrm{Co}(\succ) = \mathrm{SC}(\succ), \qquad \bigcap_{\widehat{\succ}\,\in\,\mathrm{SC}(\succ)} \mathrm{TTC}(\widehat{\succ}) \;\subseteq\; \mathcal{C}_{\mathrm{S}}(\succ) \;\subseteq\; \mathcal{C}(\succ) \;=\; \bigcup_{\widehat{\succ}\,\in\,\mathrm{SC}(\succ)} \mathrm{TTC}(\widehat{\succ}).$$

It follows from Proposition 1 that the elements of $\mathcal{C}(\succ)$ can be constructed by the sequential application of the algorithms SC and TTC to the preference profile $\succ$.

Moreover, $\mathcal{C}(\succ)$ is single-valued if and only if $\mathrm{TTC}(\widehat{\succ}) = \mathrm{TTC}(\widehat{\succ}^*)$ for all $\widehat{\succ}, \widehat{\succ}^* \in \mathrm{Co}(\succ)$. In this case, the core and the strong core coincide. Hence, if the strong core is an empty set, then the core has more than one element (Example 1 shows that the reciprocal does not hold).

From Proposition 1 we can obtain the following property of monotonicity of the core.

Corollary 1. *The core* $\mathcal{C}(\succ)$ *weakly increases with the degree of incompleteness of preferences. That is, given preference profiles* $\succ = (\succ_i)_{i\in I}$ *and* $\succ' = (\succ'_i)_{i\in I}$, *we have that*

$$[\, M(\succ_i) \subseteq M(\succ'_i), \text{ for all } i \in I\,] \qquad \Longrightarrow \qquad \mathcal{C}(\succ) \subseteq \mathcal{C}(\succ').$$

Proof. Suppose that $M(\succ_i) \subseteq M(\succ'_i)$ for all $i \in I$. It follows from the definition of the SC algorithm that $\mathrm{SC}(\succ) \subseteq \mathrm{SC}(\succ')$. Hence, Proposition 1 ensures that $\mathcal{C}(\succ) \subseteq \mathcal{C}(\succ')$. □

Despite what happens with the core, it is not clear how the strong core varies as the degree of incompleteness increases. To illustrate it, we carry out a comparative statics exercise in which incompleteness is added sequentially to agents' preferences.

Example 4. In any Shapley-Scarf housing market with three agents, the feasible allocations are

$$\begin{aligned}
\mu_1 &= [(1,h_1),(2,h_2),(3,h_3)], & \mu_2 &= [(1,h_1),(2,h_3),(3,h_2)],\\
\mu_3 &= [(1,h_2),(2,h_1),(3,h_3)], & \mu_4 &= [(1,h_2),(2,h_3),(3,h_1)],\\
\mu_5 &= [(1,h_3),(2,h_1),(3,h_2)], & \mu_6 &= [(1,h_3),(2,h_2),(3,h_1)].
\end{aligned}$$

We will use this notation to report the *core* and the *strong core* for a sequence of housing markets in which the degree of preference incompleteness increases:[11]

The two examples above show that the strong core may shrink as the incompleteness of preferences increases. To some extent, this result is not surprising. In fact, it could be thought that the likelihood of finding coalitions that weakly block a housing allocation increases as preferences become more incomplete. This is because there are greater chances that agents who cannot effectively compare houses will participate in a blocking coalition. However, the strong core does not always behave in this way:

[11]The core is computed by applying the TTC algorithm to the completions of preference profiles (see Proposition 1). In Appendix B we provide details on the strategies that can be implemented to compute the strong core.

Nicolás Leiva Díaz developed an application that computes the core and the strong core of a housing market with incomplete preferences: https://nleivad.shinyapps.io/apphousingmarketincompletepreferences/.

Preference Profile	Core and Strong Core
$h_1 \succ_1 h_2,\quad h_1 \succ_1 h_3,\quad h_2 \otimes_1 h_3$ $h_1 \succ_2 h_3 \succ_2 h_2,\quad h_1 \succ_3 h_2 \succ_3 h_3$	$\mathcal{C}_{\mathrm{S}} = \{\mu_2\}$ $\mathcal{C} = \{\mu_2\}$
$h_1 \otimes_1 h_2,\quad h_1 \succ_1 h_3,\quad h_2 \otimes_1 h_3$ $h_1 \succ_2 h_3 \succ_2 h_2,\quad h_1 \succ_3 h_2 \succ_3 h_3$	$\mathcal{C}_{\mathrm{S}} = \emptyset$ $\mathcal{C} = \{\mu_2, \mu_3\}$

Preference Profile	Core and Strong Core
$h_2 \succ_1 h_3 \succ_1 h_1$ $h_1 \succ_2 h_3 \succ_2 h_2$ $h_1 \succ_3 h_2 \succ_3 h_3$	$\mathcal{C}_{\mathrm{S}} = \{\mu_3\}$ $\mathcal{C} = \{\mu_3\}$
$h_2 \succ_1 h_1,\quad h_3 \succ_1 h_1,\quad h_2 \otimes_1 h_3$ $h_1 \succ_2 h_3 \succ_2 h_2$ $h_1 \succ_3 h_2 \succ_3 h_3$	$\mathcal{C}_{\mathrm{S}} = \emptyset$ $\mathcal{C} = \{\mu_3, \mu_6\}$
$h_2 \succ_1 h_1,\quad h_3 \succ_1 h_1,\quad h_2 \otimes_1 h_3$ $h_1 \succ_2 h_2,\quad h_3 \succ_2 h_2,\quad h_1 \otimes_2 h_3$ $h_1 \succ_3 h_2 \succ_3 h_3$	$\mathcal{C}_{\mathrm{S}} = \{\mu_4\}$ $\mathcal{C} = \{\mu_3, \mu_4, \mu_6\}$
$h_2 \succ_1 h_1,\quad h_3 \succ_1 h_1,\quad h_2 \otimes_1 h_3$ $h_1 \succ_2 h_2,\quad h_3 \succ_2 h_2,\quad h_1 \otimes_2 h_3$ $h_1 \succ_3 h_3,\quad h_2 \succ_3 h_3,\quad h_1 \otimes_3 h_2$	$\mathcal{C}_{\mathrm{S}} = \{\mu_4, \mu_5\}$ $\mathcal{C} = \{\mu_2, \mu_3, \mu_4, \mu_5, \mu_6\}$
$h_3 \otimes_1 h_1,\quad h_2 \otimes_1 h_1,\quad h_2 \otimes_1 h_3$ $h_1 \succ_2 h_2,\quad h_3 \otimes_2 h_2,\quad h_1 \otimes_2 h_3$ $h_1 \succ_3 h_3,\quad h_2 \otimes_3 h_3,\quad h_1 \otimes_3 h_2$	$\mathcal{C}_{\mathrm{S}} = \{\mu_2, \mu_4, \mu_5\}$ $\mathcal{C} = \{\mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \mu_6\}$

What ends up happening in these examples is related to a factor that is less evident: adding incompleteness *at the top* of preferences may prevent the existence of agents that can improve their situation. In other words, in the last examples the strong core increases because the number of coalitions with at least one agreement in which some member is strictly better off decreases. □

Corollary 2. *Let $[I, H, (\succ_i)_{i\in I}]$ be a Shapley-Scarf housing market in which only agent i has incomplete preferences. If the core is not single-valued, then the following properties hold:*

(a) *There are no two core allocations in which i receives the same house.*

(b) *Agent i cannot compare the houses that she receives in core allocations.*

Proof. To prove that (a) holds, let $\mu, \eta \in \mathcal{C}(\succ)$ be such that $\mu(i) = \eta(i)$. Since $\succ_{-i} = (\succ_j)_{j\neq i}$ are complete preferences, it follows from Proposition 1 that there are completions $\widehat{\succ}_i^{\mu}$ and $\widehat{\succ}_i^{\eta}$ of the preferences of agent i such that $\mathrm{TTC}(\widehat{\succ}_i^{\mu}, \succ_{-i})(i) = \mu(i) = \eta(i) = \mathrm{TTC}(\widehat{\succ}_i^{\eta}, \succ_{-i})(i)$. Since the TTC mechanism is group strategy-proof in the domain of complete and strict preferences (see Bird (1984), Moulin (1995, Lemma 3.3)), it is non-bossy in that domain (see Pápai (2000)). Hence, μ and η coincide.

To prove that (b) holds, suppose that there are allocations $\mu, \eta \in \mathcal{C}(\succ)$ such that $\mu(i) \succ_i \eta(i)$. Hence, Proposition 1 implies that there are completions $\widehat{\succ}_i^{\mu}$ and $\widehat{\succ}_i^{\eta}$ of the preferences of agent i such that $\mathrm{TTC}(\widehat{\succ}_i^{\mu}, \succ_{-i})(i) = \mu(i) \succ_i \eta(i) = \mathrm{TTC}(\widehat{\succ}_i^{\eta}, \succ_{-i})(i)$. Since $\mu(i) \succ_i \eta(i)$ implies that $\mu(i) \,\widehat{\succ}_i^{\eta}\, \eta(i)$, we obtain a contradiction with the fact that TTC is *strategy-proof* in the domain of complete and strict preferences (see Roth (1982)). □

Example 1 demonstrates that Corollary 2 does not hold when more than one agent has incomplete preferences. Moreover, when only one agent has incomplete preferences, the other agents may receive different and comparable houses in two core allocations (see Example 4).

Remark 1 [*On the role of transaction costs in models with complete and weak preferences*]

In Shapley-Scarf housing markets in which agents have complete, transitive, and weak preferences, the *strict core* may be an empty set (cf., Shapley and Scarf (1974)). Intuitively, this situation is generated by the willingness of agents to swap houses that they consider indifferent.

Our results for markets with incomplete preferences guarantee that the strict core of a market with complete and weak preferences is non-empty under transaction costs. Formally, let $[I, H, (u^i)_{i\in I}]$ be a housing market in which every agent i has a utility function $u^i : H \to \mathbb{R}$. If house-switching members of blocking coalitions must pay a positive transaction cost, then no agent will participate in a deviation to exchange houses that she considers indifferent. Also, depending on the magnitude of transaction costs, there may be agents that refrain to participate in a blocking coalition even when they know that they will receive a strictly preferred house after the deviation. Hence, the strict core of $[I, H, (u^i)_{i\in I}]$ contains the core of the housing market $[I, H, (\succ_i)_{i\in I}]$ in which there is no transaction costs and agents have potentially incomplete preferences characterized by $h \succ_i h'$ if and only if $u^i(h) > u^i(h')$. Therefore, our Proposition 1 implies that $[I, H, (u^i)_{i\in I}]$ has a non-empty strict core. □

4. On efficiency and ex-post stability

In the presence of agents with incomplete preferences, there are two natural concepts of efficiency depending on whether a redistribution of houses may leave some agents with an alternative that is incomparable with the original. These concepts, which we refer to as *weak efficiency* and *efficiency*, are equivalent to Pareto efficiency when preferences are complete.

More formally, given a housing market $[I, H, (\succ_i)_{i\in I}]$ and an allocation $\mu \in \mathcal{A}$, we will say that

- μ is *weakly efficient* when it cannot be blocked by the grand coalition I.
- μ is *efficient* when it cannot be weakly blocked by the grand coalition I.

Notice that, for the preference profiles $\succ \in \mathcal{P}^n$ that induce transitive relations $(\otimes_i)_{i\in I}$, efficiency is analogous to Pareto efficiency whenever incomparability is equiparable with indifference.

Although any allocation in the core is weakly efficient and any allocation in the strong core is efficient, the next example shows that there are housing markets in which no allocation in the core is efficient (when this occurs, the strong core is empty).[12]

Example 5. Let $[I, H, (\succ_i)_{i\in I}]$ be a housing market with three agents in which preferences are given by

$$\succ_1 :\ h_3 \succ_1 h_2 \succ_1 h_1; \qquad \succ_2 :\ h_3 \succ_2 h_1 \succ_2 h_2; \qquad \succ_3 :\ h_1 \succ_3 h_3,\ \ h_2 \succ_3 h_3,\ \ h_1 \otimes_3 h_2.$$

It follows from Proposition 1 that $\mathcal{C}(\succ) = \{[(1, h_3), (2, h_2), (3, h_1)],\ [(1, h_1), (2, h_3), (3, h_2)]\}$.

Notice that, the grand coalition can weakly block the housing allocation $[(1, h_3), (2, h_2), (3, h_1)]$ to implement $[(1, h_3), (2, h_1), (3, h_2)]$, and it can also weakly block $[(1, h_1), (2, h_3), (3, h_2)]$ to implement $[(1, h_2), (2, h_3), (3, h_1)]$. Thus, no housing allocation in $\mathcal{C}(\succ)$ is efficient. □

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[12] In Example 2, the strong core is empty and the core allocation $[(1, h_3), (2, h_2), (3, h_1)]$ is efficient. Hence, the non-existence of efficient core allocations is only sufficient to guarantee that the strong core is an empty set.

Remark 2 [*On the ex-post stability of coalitionally stable allocations*]

Let $\mathcal{C}(\succ,\mu)$ and $\mathcal{C}_{\mathrm{S}}(\succ,\mu)$ be the core and the strong core when endowments are determined by $\mu\in\mathcal{A}$. Hence, if $e\in\mathcal{A}$ satisfies $e(i)=h_i$ for all $i\in I$, then $\mathcal{C}(\succ)\equiv\mathcal{C}(\succ,e)$ and $\mathcal{C}_{\mathrm{S}}(\succ)\equiv\mathcal{C}_{\mathrm{S}}(\succ,e)$.

A housing allocation μ is *stable* when no coalition blocks it *ex-post*, in the sense that $\mu\in\mathcal{C}(\succ,\mu)$. Like in Shapley-Scarf markets with complete and transitive preferences (see Roth and Postlewaite (1977)), in our framework stability coincides with weak efficiency:

- If there exists a coalition C that blocks μ through an agreement $\sigma: C\to\mu(C)$, then the grand coalition I blocks μ through the agreement $\tilde{\sigma}: I\to H$ characterized by $(\tilde{\sigma}(i),\tilde{\sigma}(j))=(\sigma(i),\mu(j))$ for all $(i,j)\in C\times(I\setminus C)$. Therefore, weak efficiency implies stability.
- Since each allocation in $\mathcal{C}(\succ,\mu)$ is weakly efficient, stability implies weak efficiency.

As a consequence, any allocation in the core $\mathcal{C}(\succ)$ is stable.

A housing allocation μ is *strongly stable* as long as $\mu\in\mathcal{C}_{\mathrm{S}}(\succ,\mu)$. By analogous arguments to those made above, we can show that strong stability coincides with efficiency. In particular, any allocation in the strong core is strongly stable. □

5. Mechanism design under incompleteness

In this section, we characterize mechanisms that implement allocations in either the core or the strong core without incentivizing agents to misreport preferences.

Given a set of agents $I=\{1,\dots,n\}$ and a set of houses $H=\{h_1,\dots,h_n\}$, where h_i is owned by i, denote by $\mathcal{P}$ the set of (potentially) incomplete, transitive, and strict preferences defined on H.

For a preference domain $\mathcal{D}\subseteq\mathcal{P}^n$, a *mechanism* is a function $\Omega:\mathcal{D}\to\mathcal{A}$ that associates to each preference profile $(\succ_i)_{i\in I}\in\mathcal{D}$ a housing allocation of the Shapley-Scarf housing market $(I,H,(\succ_i)_{i\in I})$. Given a mechanism $\Omega:\mathcal{D}\to\mathcal{A}$, consider the following properties:

- Ω is *core-selecting* whenever $\Omega(\succ)\in\mathcal{C}(\succ)$ for any $\succ\in\mathcal{D}$.
- Ω is *strong core-selecting* whenever $\Omega(\succ)\in\mathcal{C}_{\mathrm{S}}(\succ)$ for any $\succ\in\mathcal{D}$.
- Ω is *strategy-proof* as long as—independently of the preferences of other individuals—no agent has incentives to misreport preferences when Ω is implemented. That is, there is no agent i such that, for some $\succ=(\succ_j)_{j\in I}\in\mathcal{D}$ and $\succ'_i\in\mathcal{P}$, it holds that $(\succ'_i,\succ_{-i})\in\mathcal{D}$ and
$$\Omega(\succ'_i,\succ_{-i})(i)\ \succ_i\ \Omega(\succ)(i),$$
where $\succ_{-i}=(\succ_j)_{j\neq i}$ are the preferences of agents in $I\setminus\{i\}$.[13]
- Ω is *weakly group strategy-proof* as long as—independently of the preferences of other individuals—no group of agents can misreport preferences to improve the well-being of all of its members. That is, there is no coalition C such that, for some preference profiles $\succ=(\succ_j)_{j\in I}\in\mathcal{D}$ and $\succ'_C=(\succ'_j)_{j\in C}\in\mathcal{P}^{|C|}$ it holds that $(\succ'_C,\succ_{-C})\in\mathcal{D}$ and
$$\Omega(\succ'_C,\succ_{-C})(i)\ \succ_i\ \Omega(\succ)(i),\qquad\forall i\in C,$$
where $\succ_{-C}=(\succ_j)_{j\notin C}$ are the preferences of agents in $I\setminus C$.
- Ω is *group strategy-proof* as long as—independently of the preferences of other individuals—no group of agents has incentives to misreport preferences when Ω is implemented. That is, there is no coalition C such that, for some preference profiles $\succ=(\succ_j)_{j\in I}\in\mathcal{D}$ and $\succ'_C=(\succ'_j)_{j\in C}\in\mathcal{P}^{|C|}$ it holds that $(\succ'_C,\succ_{-C})\in\mathcal{D}$ and

[13] When $\mathcal{D}=\prod_{i\in I}\mathcal{P}_i$, with $\mathcal{P}_i\subseteq\mathcal{P}$ for all $i\in I$, a mechanism $\Omega:\mathcal{D}\to\mathcal{A}$ is strategy-proof if and only if the truthful revelation of preferences is a dominant strategy equilibrium in the game in which the allocation $\Omega((\succ_i)_{i\in I})$ is implemented when each agent i reports preferences $\succ_i\in\mathcal{P}_i$.

(i) For each agent $i \in C$, $\Omega(\succ'_C, \succ_{-C})(i) \succ_i \Omega(\succ)(i)$ or $\Omega(\succ'_C, \succ_{-C})(i) = \Omega(\succ)(i)$.
(ii) There exists $i \in C$ such that $\Omega(\succ'_C, \succ_{-C})(i) \succ_i \Omega(\succ)(i)$.

- Ω is *individually rational* when there is no $i \in I$ such that $h_i \succ_i \Omega(\succ)(i)$ for some $\succ \in \mathcal{D}$.

As a consequence of Proposition 1, a mechanism $\Omega : \mathcal{P}^n \to \mathcal{A}$ is core-selecting if and only if, for each preference profile $\succ$, $\Omega(\succ)$ can be obtained by the application of TTC to a completion of $\succ$.

The following example illustrates that the way in which these completions are chosen may incentivize agents to misreport preferences.

Example 6. Consider a Shapley-Scarf housing market with four agents and let $\succ = (\succ_i)_{i\in I}$ be the preference profile characterized by

$$\succ_1:\ h_4 \succ_1 h_3 \succ_1 h_1,\ h_2 \succ_1 h_3 \succ_1 h_1,\ h_2 \otimes_1 h_4; \qquad \succ_3:\ h_1 \succ_3 h_2 \succ_3 h_3 \succ_3 h_4;$$

$$\succ_2:\ h_4 \succ_2 h_2 \succ_2 h_1,\ h_4 \succ_2 h_3 \succ_2 h_1,\ h_3 \otimes_2 h_2; \qquad \succ_4:\ h_1 \succ_4 h_4 \succ_4 h_3 \succ_4 h_2.$$

Applying the TTC algorithm to the completions of $\succ$, it follows from Proposition 1 that the correspondence between the elements of $\text{Co}(\succ)$ and the housing allocations in $\mathcal{C}(\succ)$ is given by:

$\widehat{\succ} \in \text{Co}(\succ)$	$\text{TTC}(\widehat{\succ})$
$h_4 \,\widehat{\succ}_1\, h_2 \,\widehat{\succ}_1\, h_3 \,\widehat{\succ}_1\, h_1,\ \ h_4 \,\widehat{\succ}_2\, h_2 \,\widehat{\succ}_2\, h_3 \,\widehat{\succ}_2\, h_1,\ \ \widehat{\succ}_3 = \succ_3,\ \ \widehat{\succ}_4 = \succ_4$	$\mu_1 = [(1,h_4),(2,h_2),(3,h_3),(4,h_1)]$
$h_2 \,\widehat{\succ}_1\, h_4 \,\widehat{\succ}_1\, h_3 \,\widehat{\succ}_1\, h_1,\ \ h_4 \,\widehat{\succ}_2\, h_2 \,\widehat{\succ}_2\, h_3 \,\widehat{\succ}_2\, h_1,\ \ \widehat{\succ}_3 = \succ_3,\ \ \widehat{\succ}_4 = \succ_4$	$\mu_3 = [(1,h_2),(2,h_4),(3,h_3),(4,h_1)]$
$h_4 \,\widehat{\succ}_1\, h_2 \,\widehat{\succ}_1\, h_3 \,\widehat{\succ}_1\, h_1,\ \ h_4 \,\widehat{\succ}_2\, h_3 \,\widehat{\succ}_2\, h_2 \,\widehat{\succ}_2\, h_1,\ \ \widehat{\succ}_3 = \succ_3,\ \ \widehat{\succ}_4 = \succ_4$	$\mu_2 = [(1,h_4),(2,h_3),(3,h_2),(4,h_1)]$
$h_2 \,\widehat{\succ}_1\, h_4 \,\widehat{\succ}_1\, h_3 \,\widehat{\succ}_1\, h_1,\ \ h_4 \,\widehat{\succ}_2\, h_3 \,\widehat{\succ}_2\, h_2 \,\widehat{\succ}_2\, h_1,\ \ \widehat{\succ}_3 = \succ_3,\ \ \widehat{\succ}_4 = \succ_4$	$\mu_3 = [(1,h_2),(2,h_4),(3,h_3),(4,h_1)]$

Let $\Omega : \mathcal{P}^4 \to \mathcal{A}$ be a core-selecting mechanism such that $\Omega(\succ) = \mu_1$ and $\Omega(\succ_{-2}, \succ'_2) = \mu_3$, where $h_4 \succ'_2 h_2 \succ'_2 h_3 \succ'_2 h_1$. Since agent 2 prefers h_4 to h_2 under $\succ_2$, she has incentives to misreport her preferences when Ω is implemented and preferences are $\succ$. Thus, Ω is not strategy-proof.

Since Ω is core-selecting, it follows from the table above that

$$\Omega(\succ_1, \succ_2, \succ_3, \succ_4) = \text{TTC}(\succ'_1, \succ'_2, \succ_3, \succ_4), \qquad \Omega(\succ_1, \succ'_2, \succ_3, \succ_4) = \text{TTC}(\succ^*_1, \succ'_2, \succ_3, \succ_4)$$

where $\succ'_1$ and $\succ^*_1$ are characterized by $h_4 \succ'_1 h_2 \succ'_1 h_3 \succ'_1 h_1$ and $h_2 \succ^*_1 h_4 \succ^*_1 h_3 \succ^*_1 h_1$.

Therefore, what is happening in this example is that the completions that are chosen for agent $i = 1$ do not only depend on her reported preferences. $\square$

Let $\mathcal{Q} \subsetneq \mathcal{P}$ be the set of complete, transitive, and strict preferences defined on H.

Denote by $\mathcal{G}$ the set of functions $g : \mathcal{P} \to \mathcal{Q}$ such that $g(\succ_i)$ is a completion of $\succ_i$.

It is well-known that the TTC mechanism is core-selecting and group strategy-proof in the preference domain $\mathcal{Q}^n$ (cf., Shapley and Scarf (1974), Bird (1984), Moulin (1995, Lemma 3.3)). This classical result jointly with our Proposition 1 allow us to find a family of core-selecting and group strategy-proof mechanisms defined on scenarios where agents may have incomplete preferences.

Theorem 1. *Let $\Omega : \mathcal{P}^n \to \mathcal{A}$ be a mechanism such that, for some functions $g_1, \dots, g_n \in \mathcal{G}$,*

$$\Omega(\succ_1, \dots, \succ_n) = \text{TTC}(g_1(\succ_1), \dots, g_n(\succ_n)).$$

Then, Ω is core-selecting and group strategy-proof.

Proof. The mechanism Ω is core-selecting since it associates to every preference profile $\succ \in \mathcal{P}^n$ the result of the TTC algorithm applied to a completion of $\succ$ (see Proposition 1).

Suppose that Ω is not group strategy-proof. Hence, there is a coalition C such that, for some preference profiles $\succ = (\succ_i)_{i\in I} \in \mathcal{P}^n$ and $(\succ'_i)_{i\in C} \in \mathcal{P}^{|C|}$, the following conditions hold:

- For each $i \in C$, $\Omega(\succ'_C, \succ_{-C})(i) \succ_i \Omega(\succ)(i)$ or $\Omega(\succ'_C, \succ_{-C})(i) = \Omega(\succ)(i)$.
- There exists $i \in C$ such that $\Omega(\succ'_C, \succ_{-C})(i) \succ_i \Omega(\succ)(i)$.

Since the functions $g_1, \dots, g_n$ belong to $\mathcal{G}$, we have that $P \equiv (P_i)_{i\in I} = (g_i(\succ_i))_{i\in I} \in \text{Co}(\succ) \subseteq \mathcal{Q}^n$ and $P'_C \equiv (P'_i)_{i\in C} = (g_i(\succ'_i))_{i\in C} \in \mathcal{Q}^{|C|}$.

Thus, the definition of Ω and the properties above imply that:

- For each $i \in C$, either $\text{TTC}(P'_C, P_{-C})(i)\ P_i\ \text{TTC}(P)(i)$ or $\text{TTC}(P'_C, P_{-C})(i) = \text{TTC}(P)(i)$, where $P_{-C} = (P_j)_{j\in I\setminus C}$.
- There exists $i \in C$ such that $\text{TTC}(P'_i, P_{-i})(i)\ P_i\ \text{TTC}(P)(i)$.

We conclude that the TTC mechanism is not group strategy-proof in $\mathcal{Q}^n$, a contradiction. □

As a consequence of Theorem 1, a mechanism that transforms the preferences $\succ_i$ reported by an agent i into a strict linear order using a *common* protocol $g : \mathcal{P} \to \mathcal{Q}$—and afterwards applies the algorithm TTC to the profile $(g(\succ_i))_{i\in I}$—is core-selecting and strategy-proof. However, this common protocol g cannot be induced by a ranking of houses, in the sense that $a \otimes_i b$ becomes $a \succ_i b$ whenever a is better ranked than b. Indeed, in contrast with what happens in housing markets with weak preferences (Elhers (2014)), transforming incomparabilities using a ranking may compromise the transitivity of the completion obtained. For instance, assume that $h \otimes_i h'$, $h \otimes_i h''$, and $h' \succ_i h''$. If we use a ranking of houses to transform incomparabilities into strict preferences, when h has a better ranking than h', the only completion is $h \mathrel{\widehat{\succ}_i} h' \mathrel{\widehat{\succ}_i} h''$. Hence, it is impossible to induce a completion using the ranking $h'' > h > h'$.

Given a mechanism $\Omega : \mathcal{P}^n \to \mathcal{A}$, we will say that Ω is *strongly group strategy-proof* as long as there is no coalition C such that, for some preference profiles $\succ \in \mathcal{P}^n$ and $\succ'_C \in \mathcal{P}^{|C|}$, it holds that

(i) There is no agent $i \in C$ such that $\Omega(\succ)(i) \succ_i \Omega(\succ'_C, \succ_{-C})(i)$.
(ii) There exists $i \in C$ such that $\Omega(\succ'_C, \succ_{-C})(i) \succ_i \Omega(\succ)(i)$.

In other words, Ω is strongly group strategy-proof when, independently of the preferences of other individuals, no group of agents has incentives to misreport preferences in order to improve the well-being of at least one of its members without worsening the situation of the others.

Remark 3 [*No core-selecting mechanism is strongly group strategy-proof*]

Consider a Shapley-Scarf housing market with n agents in which only agent 3 has incomplete preferences and $\succ = (\succ_i)_{i\in\{1,\dots,n\}}$ satisfies the following conditions:

$$\begin{aligned}
\succ_1:&\quad h_3 \succ_1 h_1 \succ_1 \cdots,\\
\succ_2:&\quad h_3 \succ_2 h_2 \succ_2 \cdots,\\
\succ_3:&\quad h_1 \succ_3 h_3 \succ_3 h_4 \succ_3 \cdots \succ_3 h_n, \quad h_2 \succ_3 h_3, \quad h_1 \otimes_3 h_2,\\
\succ_i:&\quad h_{i+1} \succ_i h_i \succ_i \cdots \qquad \forall i \in \{4, \dots, n-1\},\\
\succ_n:&\quad h_4 \succ_n h_n \succ_n \cdots.
\end{aligned}$$

It follows from Proposition 1 that the TTC mechanism can be applied to the completions of $\succ$ to verify that the housing allocations $\mu_1 = [(1,h_1),(2,h_3),(3,h_2),(4,h_5),\ldots,(n-1,h_n),(n,h_4)]$ and $\mu_2 = [(1,h_3),(2,h_2),(3,h_1),(4,h_5),\ldots,(n-1,h_n),(n,h_4)]$ are the only elements of the core $\mathcal{C}(\succ)$.

As a consequence, given any core-selecting mechanism $\Omega : \mathcal{P}^n \to \mathcal{A}$, there are two possibilities:

- If $\Omega(\succ) = \mu_1$, then agent 1 improves her situation and agent 3 is not worst when the coalition $\{1,3\}$ reports preferences $(\succ_1, \succ'_3)$, where $\succ'_3$ is a complete preference satisfying $h_1 \succ'_3 h_2 \succ'_3 h_3 \succ'_3 \cdots$. Indeed, $\Omega(\succ_{-3}, \succ'_3) = \mu_2$ since the core satisfies $\mathcal{C}(\succ_{-3}, \succ'_3) = \{\mu_2\}$.
- If $\Omega(\succ) = \mu_2$, then agent 2 improves her situation and agent 3 is not worst when the coalition $\{2,3\}$ reports preferences $(\succ_2, \succ^*_3)$, where $\succ'_3$ is a complete preference satisfying $h_2 \succ^*_3 h_1 \succ^*_3 h_3 \succ^*_3 \cdots$. Indeed, $\Omega(\succ_{-3}, \succ^*_3) = \mu_1$ since $\mathcal{C}(\succ_{-3}, \succ^*_3) = \{\mu_1\}$.

We conclude that no core-selecting mechanism $\Omega : \mathcal{P}^n \to \mathcal{A}$ is strongly group strategy-proof. □

Let $\mathcal{R} \subseteq \mathcal{P}$ be the set of preferences that induce *transitive* incomparability relations. If $\widetilde{\mathcal{R}}$ is the set of complete, transitive, and weak preferences defined on H, then $\mathcal{R}$ and $\widetilde{\mathcal{R}}$ and can be identified through the bijection $\tau : \mathcal{R} \to \widetilde{\mathcal{R}}$ characterized by

$$h\,\tau(\succ_i)\,h' \qquad \Longleftrightarrow \qquad [\,h \succ_i h' \quad \text{or} \quad h = h' \quad \text{or} \quad h \otimes_i h'\,].$$

In other words, under $\tau(\succ_i)$ a house h is *at least as preferred* as another house h' when $h' \succ_i h$ does not hold. As a consequence, h and h' are *indifferent* under $\tau(\succ_i)$ if and only if $h \otimes_i h'$.

Notice that, for any preference profile $\succ = (\succ_i)_{i\in I} \in \mathcal{R}^n$ and housing allocation $\mu \in \mathcal{A}$,

- $\mathcal{C}_{\mathrm{S}}(\succ)$ coincides with the *strict core* of the Shapley-Scarf housing market $[I, H, (\tau(\succ_i))_{i\in I}]$.[14]
- μ efficient in $[I, H, (\succ_i)_{i\in I}]$ if and only if it is Pareto efficient in $[I, H, (\tau(\succ_i))_{i\in I}]$.

Let $\mathcal{D}^* \subseteq \mathcal{P}^n$ be the set of preference profiles $\succ = (\succ_i)_{i\in I}$ such that the strong core $\mathcal{C}_{\mathrm{S}}(\succ)$ is non-empty. It follows from Examples 1, 2, and 4 that $\mathcal{Q}^n \subsetneq \mathcal{D}^* \cap \mathcal{R}^n \subsetneq \mathcal{D}^* \subsetneq \mathcal{P}^n$.

Moreover, by identifying *incomparability* with *indifference*, the mapping $(\succ_i)_{i\in I} \longrightarrow (\tau(\succ_i))_{i\in I}$ determines a one-to-one correspondence between the elements in $\mathcal{D}^* \cap \mathcal{R}^n$ and the complete, transitive, and weak preference profiles for which the *strict core* is non-empty.[15]

Alcalde-Unzu and Molis (2011) and Jaramillo and Manjunath (2012) generalize the TTC algorithm to account for the presence of indifferences in Shapley-Scarf housing markets with complete preferences. The algorithms that they introduce—*Top Trading Absorbing Sets Mechanisms* (TTAS) and *Top Cycles Mechanisms* (TC)—are weakly group strategy-proof in $\widetilde{\mathcal{R}}^n$ and implement an allocation in the strict core when it is non-empty (see Aziz and Keijzer (2012), Ahmad (2021)).[16] We will appeal to these results to characterize incentive properties of strong-core-selecting mechanisms.

[14]The *strict core* of $[I, H, (\tau(\succ_i))_{i\in I}]$ is the set of housing allocations $\mu \in \mathcal{A}$ for which there is *no* coalition C such that, for some agreement σ among its members, the following conditions hold: (i) for all $i \in C$, the house $\sigma(i)$ is at least as preferred as $\mu(i)$ under $\tau(\succ_i)$; and (ii) there exists $i \in C$ such that $\sigma(i)$ is strictly preferred to $\mu(i)$ under $\tau(\succ_i)$.

[15]The necessary and sufficient conditions determined by Quint and Wako (2004) to characterize the preference profiles in which the strict core is non-empty could be adapted to our framework to characterize $\mathcal{D}^* \cap \mathcal{R}^n$.

[16]The mechanisms introduced by Alcalde-Unzu and Molis (2011) and Jaramillo and Manjunath (2012) are based on algorithms that require choosing between cycles composed of agents and houses. This is done using a *prioritization* of I or H. To simplify the notation, the symbols TTAS and TC do not make reference to these underlying priority rankings.

Theorem 2. *The mechanisms* $\Omega_1, \Omega_2 : \mathcal{R}^n \to \mathcal{A}$ *characterized by*

$$\Omega_1(\succ) = \text{TTAS}(\tau(\succ_1), \dots, \tau(\succ_n)) \qquad \textit{and} \qquad \Omega_2(\succ) = \text{TC}(\tau(\succ_1), \dots, \tau(\succ_n))$$

are efficient, individually rational, and weakly group strategy-proof. Moreover, these mechanisms select allocations in the strong core when it is non-empty.

Proof. Let $\succ = (\succ_i)_{i\in I} \in \mathcal{R}^n$. Since $\succ$ induces transitive incomparability relations, the set of efficient and individually rational allocations of $[I, H, \succ]$ coincides with the collection of Pareto efficient and individually rational allocations of $[I, H, (\tau(\succ_i))_{i\in I}]$. On the other hand, Corollary 1 and Theorem 2 in Alcalde-Unzu and Molis (2011) and Proposition 3 in Jaramillo and Manjunath (2012) ensure that the housing allocations $\text{TTAS}((\tau(\succ_i))_{i\in I})$ and $\text{TC}((\tau(\succ_i))_{i\in I})$ are Pareto efficient and individually rational. Therefore, the mechanisms Ω_1 and Ω_2 are efficient and individually rational.

Since $\mathcal{C}_{\text{S}}(\succ)$ coincides with the strict core of $[I, H, (\tau(\succ_i))_{i\in I}]$, the results of Alcalde-Unzu and Molis (2011, Theorem 4) and Aziz and Keijzer (2012, Corollary 1) ensure that $\Omega_i(\succ) \in \mathcal{C}_{\text{S}}(\succ)$ for each $i \in \{1,2\}$.

If Ω_1 is not weakly group strategy-proof in $\mathcal{R}^n$, then there exist $C \subseteq I$, $\succ = (\succ_i)_{i\in I} \in \mathcal{R}^n$, and $(\succ'_i)_{i\in C} \in \mathcal{R}^{|C|}$ such that $\Omega_1(\succ'_C, \succ_{-C})(i) \succ_i \Omega_1(\succ)(i)$ for all $i \in C$. Hence, when preferences are $(\tau(\succ_i))_{i\in C}$, each $i \in C$ strictly prefers $\text{TTAS}((\tau(\succ'_j))_{j\in C}, (\tau(\succ_j))_{j\notin C})(i)$ to $\text{TTAS}(\tau(\succ_1), \dots, \tau(\succ_n))(i)$. This contradicts Ahmad (2021, Proposition 2), which proves that TTAS is weakly group strategy-proof in the domain of preferences $\widetilde{\mathcal{R}}^n$. Analogously, Ω_2 is weakly group strategy-proof in $\mathcal{R}^n$ as a consequence of Ahmad (2021, Proposition 3). □

Plaxton (2013) introduces a family of strategy-proof mechanisms defined on $\widetilde{\mathcal{R}}^n$ that implement allocations in the strict core when it is non-empty and reduce the computational complexity of both TTAS and TC. Analogous arguments to those made in the proof of Theorem 2 ensure that any of these mechanisms $\Gamma : \widetilde{\mathcal{R}}^n \to \mathcal{A}$ induces a strong core-selecting and strategy-proof mechanism in $\mathcal{D}^* \cap \mathcal{R}^n$ through the rule that associates to each $\succ = (\succ_i)_{i\in I}$ the housing allocation $\Gamma(\tau(\succ_1), \dots, \tau(\succ_n))$.

It follows from Theorems 1 and 2 that there are several mechanisms satisfying (weak) efficiency, individual rationality, and strategy-proofness. This result differs from what occurs in Shapley-Scarf housing markets with complete, transitive, and strict preferences, where TTC is the *only* Pareto efficient, individually rational, and strategy-proof mechanism (see Ma (1994)).

Remark 4 [*On the essential single-valuedness of the strong core*]

We will say that $\mathcal{C}_{\text{S}}(\succ)$ is *essentially single-valued* when for each agent i and for all $\mu, \eta \in \mathcal{C}_{\text{S}}(\succ)$ we have that either $\mu(i) = \eta(i)$ or $\mu(i) \otimes_i \eta(i)$. We claim that the strong core is essentially single-valued in the subdomain $\mathcal{R}^n$. Notice that, for each preference profile $\succ \in \mathcal{R}^n$:

- $\mathcal{C}_{\text{S}}(\succ)$ coincides with the strict core of the housing markets $(I, H, (\tau(\succ_i))_{i\in I})$.
- $\mathcal{C}_{\text{S}}(\succ)$ is essentially single-valued if and only if each agent considers all the allocations in the strict core of $(I, H, (\tau(\succ_i))_{i\in I})$ indifferent.

Therefore, the claim follows from Theorem 2 in Wako (1991) (cf., Ma (1994, Theorem 3) and Quint and Wako (2004, Theorem 7.4)). As a consequence, any attempt to find $\succ \in \mathcal{P}^n$ such that $\mathcal{C}_{\text{S}}(\succ)$ is not essentially single-valued must concentrate on those profiles of preferences for which the relations $(\otimes_i)_{i\in I}$ induced by $\succ$ are intransitive. □

6. Extensions to housing allocation problems

In this section we extend our results to the *housing allocation problems* introduced by Hylland and Zeckhauser (1979) and Abdulkadiroğlu and Sönmez (1999), in which there are agents without endowments and vacant houses.

Housing allocation with existing tenants. Let $[I_1, I_2, H, (\succ_i)_{i\in I_1\cup I_2}]$ be a *housing allocation problem with existing tenants and incomplete preferences* in which there is a set $H = \{h_1, \dots, h_n\}$ of houses and a set $I = I_1 \cup I_2$ of agents, where $I_1 = \{1, \dots, m\}$ is a set of *tenants* and $I_2 = \{m+1, \dots, n\}$ is a set of *newcomers*. It is assumed that h_i is occupied by agent $i \in I_1$ and that h_j is vacant for each $j > m$.

As in the previous sections, $\succ_i \subseteq H \times H$ is the set of pairs (h, h') such that h is strictly preferred to h' by agent $i \in I$. Moreover, each $\succ_i$ induces an incomplete, transitive, and strict preference for houses, and the associated relation of incomparability $\otimes_i$ is not necessarily transitive.

In this context, given a housing allocation μ we will say that

- μ is *weakly efficient* when there is no $\eta \in \mathcal{A}$ that *dominates* it, in the sense that
  - $\eta(j) \succ_j \mu(j)$ or $\eta(j) = \mu(j)$ for all $j \in I$,
  - $\eta(i) \succ_i \mu(i)$ for some $i \in I$.
- μ is *efficient* when there is no $\eta \in \mathcal{A}$ that *strongly dominates* it, in the sense that
  - $\eta(j) \succ_j \mu(j)$, $\eta(j) = \mu(j)$, or $\eta(j) \otimes_j \mu(j)$ for all $j \in I$,
  - $\eta(i) \succ_i \mu(i)$ for some $i \in I$.
- μ is *individually rational* when there is no tenant $i \in I_1$ such that $h_i \succ_i \mu(i)$.

Let $\mathcal{W}^{\text{IR}}(\succ)$ be the set of weakly efficient and individually rational housing allocations, and $\mathcal{E}^{\text{IR}}(\succ)$ be the set of efficient and individually rational allocations. As weak efficiency and efficiency coincide for any profile of complete, transitive, and strict preferences, $\mathcal{W}^{\text{IR}}(\succ) = \mathcal{E}^{\text{IR}}(\succ)$ for all $\succ \in \mathcal{Q}^n$.

Since $\mathcal{A}$ is a finite set and the *dominance relation* is strict and transitive, $\mathcal{W}^{\text{IR}}(\succ)$ is non-empty for all preference profiles $\succ \in \mathcal{P}^n$. However, as the following example illustrates, when the relations $(\otimes_i)_{i\in I_1\cup I_2}$ induced by $\succ$ are intransitive, $\mathcal{E}^{\text{IR}}(\succ)$ can be an empty set.

Example 7. Consider a housing allocation problem with three houses and three agents such that $I_1 = \{1\}$ and $I_2 = \{2, 3\}$. Assume that, for each agent i, $h_1 \succ_i h_2$, $h_1 \otimes_i h_3$, and $h_2 \otimes_i h_3$. Given a housing allocation μ, denote by $i, j, k \in I$ the agents that satisfy $\mu(i) = h_1$, $\mu(j) = h_2$, and $\mu(k) = h_3$.

Let $\eta \in \mathcal{A}$ be such that $\eta(i) = h_3$, $\eta(j) = h_1$, and $\eta(k) = h_2$. It follows that μ is strongly dominated by η. Since μ was arbitrary, we conclude that the set of efficient allocations is empty, which implies that $\mathcal{E}^{\text{IR}}(\succ)$ is an empty set too. □

Since the *strong dominance relation* is strict and transitive in the preference domain $\mathcal{R}^n$, the finiteness of $\mathcal{A}$ ensures that $\mathcal{E}^{\text{IR}}(\succ)$ is non-empty for all $\succ \in \mathcal{R}^n$.

Moreover, for any $\succ = (\succ_i)_{i\in I_1\cup I_2} \in \mathcal{R}^n$, the set $\mathcal{E}^{\text{IR}}(\succ)$ coincides with the collection of Pareto efficient and individually rational allocations of the housing allocation problem with existing tenants and weak preferences $[I_1, I_2, H, (\tau(\succ_i))_{i\in I_1\cup I_2}]$, where $\tau : \mathcal{R} \to \widetilde{\mathcal{R}}$ is the function that associates incomplete preferences with weak preferences by identifying *incomparability* with *indifference* (see Section 5).

When agents have complete, transitive, and strict preferences, Abdulkadiroğlu and Sönmez (1999) extend the Top Trading Cycles mechanism to housing allocation problems in which tenants and newcomers coexist. More precisely, given a bijective function $f : \{1, \dots, n\} \to I$ representing an *ordering of agents*, these authors introduce a mechanism $\varphi^f : \mathcal{Q}^n \to \mathcal{A}$ that associates to any preference profile the outcome of a variant of the TTC algorithm that respects the property rights of existing tenants and—at each step of the process—gives the ownership of remaining vacant houses to the unmatched agent with the highest priority under f. Since the mechanism φ^f is Pareto efficient and individually rational in the preference domain $\mathcal{Q}^n$ (cf., Abdulkadiroğlu and Sönmez (1999)), it follows that

$$\bigcup_{f \in \mathcal{F}} \varphi^f(\succ) \subseteq \mathcal{E}^{\mathrm{IR}}(\succ) = \mathcal{W}^{\mathrm{IR}}(\succ), \qquad \forall \succ \in \mathcal{Q}^n,$$

where $\mathcal{F}$ is the set of orderings of agents. Also, φ^f is group strategy proof in $\mathcal{Q}^n$ (cf., Pápai (2000)).[17]

Furthermore, in the domain of complete and transitive preferences, the mechanism TC : $\widetilde{\mathcal{R}}^n \to \mathcal{A}$ introduced by Jaramillo and Manjunath (2012) can be applied to housing allocation problems with existing tenants and always generates a Pareto efficient and individually rational allocation.

These properties allow us to adapt our previous findings to show that:

- For any preference profile $\succ \in \mathcal{P}^n$,

$$\bigcap_{\widehat{\succ} \in \mathrm{SC}(\succ)} \mathcal{W}^{\mathrm{IR}}(\widehat{\succ}) \subseteq \mathcal{E}^{\mathrm{IR}}(\succ) \subseteq \mathcal{W}^{\mathrm{IR}}(\succ) = \bigcup_{\widehat{\succ} \in \mathrm{SC}(\succ)} \mathcal{W}^{\mathrm{IR}}(\widehat{\succ}).$$

- Given an ordering f of agents and functions $g_1, \ldots, g_n \in \mathcal{G}$, the mechanism

$$\Phi(\succ_1, \ldots, \succ_n) = \varphi^f(g_1(\succ_1), \ldots, g_n(\succ_n))$$

  is weakly efficient, individually rational, and group strategy-proof in $\mathcal{P}^n$.
- The mechanism

$$\Psi(\succ_1, \ldots, \succ_n) = \mathrm{TC}(\tau(\succ_1), \ldots, \tau(\succ_n))$$

  is efficient, individually rational, and weakly group strategy-proof in $\mathcal{R}^n$.

The proof of these claims is given in Appendix C.

Housing allocation. Let $[I, H, (\succ_i)_{i \in I}]$ be a *housing allocation problem with incomplete preferences* in which there is a set $I = \{1, \ldots, n\}$ of agents, a set $H = \{h_1, \ldots, h_n\}$ of houses, and a profile $(\succ_i)_{i \in I} \in \mathcal{P}^n$ determining the (potentially) incomplete preferences of agents in I. Unlike what happens in Shapley-Scarf housing markets, in this context all houses are initially vacant.

Denote by $\mathcal{W}(\succ)$ and $\mathcal{E}(\succ)$ the sets of *weakly efficient* and *efficient* housing allocations, respectively. Since $\mathcal{A}$ is a finite set and the *dominance relation* is strict and transitive, $\mathcal{W}(\succ)$ is always non-empty. On the other hand, $\mathcal{E}(\succ)$ can be an empty set (see Example 7). However, for any preference profile $\succ = (\succ_i)_{i \in I} \in \mathcal{R}^n$, the set $\mathcal{E}(\succ)$ is non-empty and coincides with the collection of Pareto efficient allocations of the housing allocation problem $[I, H, (\tau(\succ_i))_{i \in I}]$.

In housing allocation problems with complete, transitive, and weak preferences, Svensson (1994) shows that the mechanism $\mathrm{SD}^f : \widetilde{\mathcal{R}}^n \to \mathcal{A}$ that implements the *serial dictatorship* algorithm induced by an ordering f of agents is Pareto efficient and strategy-proof (cf., Bogomolnaia, Deb, Elhers (2005)).

Moreover, any Pareto efficient allocation is the outcome of a serial dictatorship mechanism (see Svensson (1994)). In the sub-domain $\mathcal{Q}^n \subsetneq \widetilde{\mathcal{R}}^n$ of complete, transitive, and strict preferences, the mechanism SD^f becomes group strategy-proof (cf., Svensson (1999), Papai (2000), Elhers (2002)).

[17] Abdulkadiroğlu and Sönmez (1999) also introduce the mechanism *You Request My House–I Get Your Turn*, which is equivalent to φ^f (cf., Sönmez and Ünver (2005, 2010)).

These results allow us to show that the following properties hold:

- For any preference profile $\succ \in \mathcal{P}^n$,

$$\bigcap_{\widehat{\succ}\, \in \mathrm{SC}(\succ)} \bigcup_{f\in\mathcal{F}} \mathrm{SD}^f(\widehat{\succ}) \;\subseteq\; \mathcal{E}(\succ) \;\subseteq\; \mathcal{W}(\succ) \;=\; \bigcup_{\widehat{\succ}\, \in \mathrm{SC}(\succ)} \bigcup_{f\in\mathcal{F}} \mathrm{SD}^f(\widehat{\succ}),$$

  where $\mathcal{F}$ is the set of orderings of agents.
- Given an ordering f of agents and functions $g_1, \dots, g_n \in \mathcal{G}$, the mechanism

$$\Phi(\succ_1, \dots, \succ_n) = \mathrm{SD}^f(g_1(\succ_1), \dots, g_n(\succ_n))$$

  is weakly efficient and group strategy-proof in $\mathcal{P}^n$.
- Given an ordering f of agents, the mechanism

$$\Psi(\succ_1, \dots, \succ_n) = \mathrm{SD}^f(\tau(\succ_1), \dots, \tau(\succ_n))$$

  is efficient and strategy-proof in $\mathcal{R}^n$.

The proof of these claims is given in Appendix C.

For housing allocation problems, Elhers (2002) shows that Pareto efficiency is incompatible with group strategy-proofness in the domain $\widetilde{\mathcal{R}}^n$ of complete, transitive, and weak preferences. Although $\widetilde{\mathcal{R}}^n$ can be identified with the set $\mathcal{R}^n \subseteq \mathcal{P}^n$ of preference profiles that induce transitive incomparability relations, the weak efficiency and group strategy-proofness of $\Phi(\succ_1, \dots, \succ_n) = \mathrm{SD}^f(g_1(\succ_1), \dots, g_n(\succ_n))$ do not contradict Elhers' result. Indeed, weak efficiency is weaker than Pareto efficiency in $\mathcal{R}^n$.

7. Concluding Remarks

We added incomplete preferences to the housing market of Shapley and Scarf (1974) and the housing allocation problems of Hylland and Zeckhauser (1979) and Abdulkadiroğlu and Sönmez (1999).

In Shapley-Scarf housing markets, we have proposed and characterized two concepts of coalitional stability: the *core* and the *strong core*. We have shown that the core coincides with the *competitive allocations* of the markets in which agents rank houses following completions of their preferences (cf., Roth and Postlewaite (1977)). As a consequence, when blocking coalitions can only include informed individuals, the incompleteness of preferences induces indeterminacy. On the other hand, since the strong core may be an empty set, the existence of coalitionally stable solutions is compromised when those who participate in a blocking coalition do not necessarily know how to compare their situation before and after the deviation. Intuitively, the inclusion of uninformed agents in a coalition increases the blocking power of the other members.

From the perspective of incentives, we have shown that there are many core-selecting mechanisms that are group strategy-proof in the full domain of preferences. Moreover, in the domain of preferences in which incomparability relations are transitive, there exist several efficient, individually rational, and weakly group strategy-proof mechanisms that select allocation in the strong core when it is non-empty.

In the housing allocation problem introduced by Hylland and Zeckhauser (1979), we have shown that an allocation is weakly efficient if and only if it is the outcome of a serial dictatorship rule applied to some completion of agents' preferences. Thus, a mechanism that applies a serial dictatorship rule to a completion of agents' preferences is weakly efficient and group strategy-proof in the full domain of preferences. Moreover, in the domain of preferences in which the incomparability relations are transitive, we found several efficient and strategy-proof mechanisms: it is sufficient to apply a serial dictatorship rule to the complete preferences obtained by identifying incomparability with indifference.

Analogous results hold for the housing allocation problem with existing tenants introduced by Abdulkadiroğlu and Sönmez (1999). Indeed, by replacing the serial dictatorship rule with one of the extensions of the TTC algorithm introduced by these authors—any version of *You Request My House–I Get Your Turn*—we obtain a weakly efficient, individually rational, and group strategy-proof mechanism in the full domain of preferences. Furthermore, restricting preference profiles to those in which incomparabilities are transitive, we obtain an efficient, individually rational, and weakly group strategy-proof mechanism by applying any of the algorithms introduced by Jaramillo and Manjunath (2012) to the complete preferences obtained by identifying incomparability with indifference.

A direction to extend our results is to study kidney exchange when patients do not have enough information to compare all potential donors. Indeed, incompleteness of preferences appears naturally in this context, because medical tests to determine the compatibility between patients and donors only generate partial information (see Smeulders, Bartier, Crama, Spieksma (2021)). Moreover, allowing patients to report incomplete preferences may increase the number of agents in kidney exchange platforms, since the complexity inherent to the process of evaluating potential donors discourages participation (cf., Sönmez, Ünver, and Yenmez (2020)).

As Roth, Sönmez, and Ünver (2004) point out, the outside option that the cadaveric waiting list represents—the possibility of receiving a lottery instead of a kidney from a living donor—makes a kidney exchange platform more complex than a Shapley-Scarf housing market. Nevertheless, by analogy with our results, we would expect that an efficient, individually rational, and strategy-proof mechanism can be obtained by applying the *Top Trading Cycles and Chains* algorithm introduced by Roth, Sönmez, and Ünver (2004) to a completion of incomplete preferences. From a logistical point of view, it can also be interesting to study the existence of coalitionally stable or efficient allocations that minimize the size of the "trading cycles" involved in their implementation (cf., Ashlagi, Gamarnik, Rees, and Roth (2012)). These issues are left for future research.

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Appendix A – Proof of Proposition 1

Claim A1. *For every preference profile* $\succ = (\succ_i)_{i\in I}$*, we have that* $\mathrm{Co}(\succ) = \mathrm{SC}(\succ)$*.*

Proof. Since $\mathrm{SC}(\succ) \subseteq \mathrm{Co}(\succ)$, given a preference profile $\widehat{\succ} = (\widehat{\succ}_i)_{i\in I} \in \mathrm{Co}(\succ)$, we want to prove that $\widehat{\succ} \in \mathrm{SC}(\succ)$. Let $\widehat{\succ}_i^*$ be the strict linear order that is obtained by applying the SC algorithm to $\succ_i$ in the following way (which considers information about $\widehat{\succ}_i$ to take decisions at Step 2):

- **Step 1:** For every $a, b \in H$ such that $a \succ_i b$, define $a \,\widehat{\succ}_i^*\, b$. Let $Z(\succ_i) = M(\succ_i)$.
- **Step 2:** Given $(a, b) \in Z(\succ_i)$,

  Step 2.1: Define $a \,\widehat{\succ}_i^*\, b$ whenever $a \,\widehat{\succ}_i\, b$. Apply the following rules:

  (1) If there exists $c \in H$ such that $c \succ_i a$ and $c \otimes_i b$, define $c \,\widehat{\succ}_i^*\, b$.
  When such c exists, $\widehat{\succ}_i$ must also satisfy $c \,\widehat{\succ}_i\, b$. Indeed, $\widehat{\succ} \in \mathrm{Co}(\succ)$ ensures that $c \,\widehat{\succ}_i\, a$. Thus, $a \,\widehat{\succ}_i\, b$ and the transitivity of $\widehat{\succ}_i$ guarantees that $c \,\widehat{\succ}_i\, b$.

  (2) If there exists $c \in H$ such that $b \succ_i c$ and $a \otimes_i c$, define $a \,\widehat{\succ}_i^*\, c$.
  When such c exists, $\widehat{\succ}_i$ must also satisfy $a \,\widehat{\succ}_i\, c$. Indeed, $\widehat{\succ} \in \mathrm{Co}(\succ)$ ensures that $b \,\widehat{\succ}_i\, c$. Thus, $a \,\widehat{\succ}_i\, b$ and the transitivity of $\widehat{\succ}_i$ guarantees that $a \,\widehat{\succ}_i\, c$.

  Step 2.2: Define $b \,\widehat{\succ}_i^*\, a$ whenever $b \,\widehat{\succ}_i\, a$. Apply the following rules:

  (1) If there exists $c \in H$ such that $c \succ_i b$ and $c \otimes_i a$, define $c \,\widehat{\succ}_i^*\, a$.
  When such c exists, $\widehat{\succ}_i$ must also satisfy $c \,\widehat{\succ}_i\, a$. Indeed, $\widehat{\succ} \in \mathrm{Co}(\succ)$ ensures that $c \,\widehat{\succ}_i\, b$. Thus, $b \,\widehat{\succ}_i\, a$ and the transitivity of $\widehat{\succ}_i$ guarantees that $c \,\widehat{\succ}_i\, a$.

  (2) If there exists $c \in H$ such that $a \succ_i c$ and $b \otimes_i c$, define $b \,\widehat{\succ}_i^*\, c$.
  When such c exists, $\widehat{\succ}_i$ must also satisfy $b \,\widehat{\succ}_i\, c$. Indeed, $\widehat{\succ} \in \mathrm{Co}(\succ)$ ensures that $a \,\widehat{\succ}_i\, c$. Thus, $b \,\widehat{\succ}_i\, a$ and the transitivity of $\widehat{\succ}_i$ guarantees that $b \,\widehat{\succ}_i\, c$.
- **Step 3:** Eliminate from $Z(\succ_i)$ the pairs $(h_j, h_k) \in H \times H$, with $j < k$, for which it was defined in the previous step that either $h_j \,\widehat{\succ}_i\, h_k$ or $h_k \,\widehat{\succ}_i\, h_j$.
- **Step 4:** Repeat Steps 2 and 3 until $Z(\succ_i) = \emptyset$.

Since by construction we have that $\widehat{\succ}_i = \widehat{\succ}_i^*$ for each $i \in I$, it follows that the preference profile $\widehat{\succ}$ can be obtained by applying the SC algorithm. □

Claim A2. *For every preference profile* $\succ = (\succ_i)_{i\in I}$*, we have that*

$$\bigcap_{\widehat{\succ}\,\in\,\mathrm{SC}(\succ)} \mathrm{TTC}(\widehat{\succ}) \;\subseteq\; \mathcal{C}_{\mathrm{S}}(\succ) \;\subseteq\; \mathcal{C}(\succ) \;=\; \bigcup_{\widehat{\succ}\,\in\,\mathrm{SC}(\succ)} \mathrm{TTC}(\widehat{\succ}).$$

Proof. Since $\mathrm{TTC}(\widehat{\succ}) = \mathbb{K}(\widehat{\succ})$ for any $\widehat{\succ} \in \mathrm{Co}(\succ)$, it follows from Claim A1 and the definitions of the core and the strong core that it is sufficient to prove that $\mathcal{C}(\succ)$ is a subset of $\bigcup_{\widehat{\succ}\,\in\,\mathrm{SC}(\succ)} \mathrm{TTC}(\widehat{\succ})$.

Let $\mathcal{F}(C)$ be the set of agreements among the members of a coalition C. Given $\widehat{\succ} \in \mathrm{Co}(\succ)$ and $\mu \in \mathcal{A}$, denote by $\Omega_\mu(\widehat{\succ})$ the (possible empty) collection of pairs (C, σ) such that, when preferences are given by $\widehat{\succ}$, the coalition C blocks the matching μ through the agreement $\sigma \in \mathcal{F}(C)$.

Fix $\mu \in \mathcal{C}(\succ)$. We claim that, for any $\widehat{\succ} \in \mathrm{Co}(\succ)$, if $(C, \sigma) \in \Omega_\mu(\widehat{\succ})$ then there exists $i \in C$ such that $\mu(i) \otimes_i \sigma(i)$. By contradiction, suppose that every $i \in C$ can compare the houses $\mu(i)$ with $\sigma(i)$ under $(\succ_i)_{i\in I}$. Since $(C, \sigma) \in \Omega_\mu(\widehat{\succ})$, for all $i \in C$ we have that $\sigma(i) \,\widehat{\succ}_i\, \mu(i)$ or $\sigma(i) = \mu(i)$, and there exists $j \in C$ such that $\sigma(j) \,\widehat{\succ}_j\, \mu(j)$. Without loss of generality, suppose that there exists $m \in \{1, \ldots, n\}$ such that $C = \{1, \ldots, m\}$ and for some $k \in \{1, \ldots, m\}$ we have that $\sigma(i) \,\widehat{\succ}_i\, \mu(i)$, for all $i \in \{1, \ldots, k\}$ and $\sigma(i) = \mu(i)$ for all $i \in \{k+1, \ldots, m\}$. As every agent $i \in \{1, \ldots, k\}$ knows how to compare $\mu(i)$ with $\sigma(i)$ under $(\succ_i)_{i\in I}$, it must be the case that $\sigma(i) \succ_i \mu(i)$, for all $i \in \{1, \ldots, k\}$.[18] Thus, C blocks μ through the agreement σ under $\succ$, which contradicts the fact that $\mu \in \mathcal{C}(\succ)$.

[18]Indeed, $\mu(i) \succ_i \sigma(i)$ implies that $\mu(i) \,\widehat{\succ}_i\, \sigma(i)$, which is a contradiction.

Let $H_\mu(i)$ be the collection of houses that agent i can obtain by participating of a blocking coalition of μ when preferences are given by some completion of $\succ$:

$$H_\mu(i) = \left\{h \in H \,:\, \exists \widehat{\succ} \in \mathrm{Co}(\succ),\, \exists (C,\sigma) \in \Omega_\mu(\widehat{\succ}),\, i \in C,\, h = \sigma(i)\right\}.$$

Also, let $Q_\mu(i)$ be the houses in $H(i)$ that i does not know how to compare with $\mu(i)$:

$$Q_\mu(i) = \{h \in H(i) : h \otimes_i \mu(i)\}.$$

Denote by $\widehat{\succ}^* \in \mathrm{Co}(\succ)$ the preference profile such that, for each $i \in I$, $\mu(i) \,\widehat{\succ}_i^*\, h$ for all $h \in Q_\mu(i)$. Notice that a preference profile $\widehat{\succ}^*$ with these characteristics always exists. It suffices that, in the Step 2 of the application of the SC algorithm to $\succ_i$, the pairs $\{(\mu(i), h) : h \in Q_\mu(i)\}$ are chosen before any other element of $M(\succ_i)$.

We claim that $\mu \in \mathrm{TTC}(\widehat{\succ}^*)$. Since $\mathrm{TTC}(\widehat{\succ}^*) = \mathbb{K}(\widehat{\succ}^*)$, when $\mu \notin \mathrm{TTC}(\widehat{\succ}^*)$ there exists a coalition C and an agreement $\sigma \in \mathcal{F}(C)$ such that $(C,\sigma) \in \Omega_\mu(\widehat{\succ}^*)$. Since $\mu \in \mathcal{C}(\succ)$, there exists $i \in C$ such that $\sigma(i) \in Q_\mu(i)$. By the construction of $\widehat{\succ}_i^*$, it follows that $\mu(i) \,\widehat{\succ}_i^*\, \sigma(i)$. This contradicts the fact that C blocks μ through the agreement σ under $\widehat{\succ}^*$. □

Appendix B – A methodology to compute $\mathcal{C}_S(\succ)$ from $\mathcal{C}(\succ)$

Given a housing market $[I, H, (\succ_i)_{i\in I}]$ and $\mu \in \mathcal{A}$, let $I_i(\mu) = \{k \in I : h_k \succ_i \mu(i) \text{ or } h_k = \mu(i) \text{ or } h_k \otimes_i \mu(i)\}$ be the owners of the houses that agent i considers at least as good as $\mu(i)$ or incomparable with $\mu(i)$. Assuming that each agent i announces the members of $I_i(\mu)$, denote by S_μ the collection of *cycles* in which someone announces the owner of a house that she considers strictly better than her assignment under μ.[19]

It follows from the definition of the *strong core* that

$$\mu \in \mathcal{C}_{\mathrm{S}}(\succ) \qquad \Longleftrightarrow \qquad [\ \mu \in \mathcal{C}(\succ) \quad \text{and} \quad S_\mu = \emptyset\ ].$$

Furthermore, if $\overline{H}_i = \{h \in H : \nexists h' \in H,\ h' \succ_i h\}$ is the set of houses that agent i considers her best alternatives, then

$$\left[\ \forall i \in I : \ \mu(i) \in \overline{H}_i\ \right] \qquad \Longrightarrow \qquad \mu \in \mathcal{C}_{\mathrm{S}}(\succ).$$

Although these criteria are not computationally efficient, they are useful to determine the strong core when there are few agents in the market.

Example B1. Consider the Shapley-Scarf housing market described in Example 1, in which there are four agents with preferences characterized by

$\succ_1$: $h_2 \succ_1 h_3 \succ_1 h_1,\ h_4 \succ_1 h_3 \succ_1 h_1,\ h_2 \otimes_1 h_4$;

$\succ_2$: $h_1 \succ_2 h_2 \succ_2 h_3 \succ_2 h_4$;

$\succ_3$: $h_1 \succ_3 h_4 \succ_3 h_3 \succ_3 h_2$;

$\succ_4$: $h_2 \succ_4 h_4 \succ_4 h_1,\ h_2 \succ_4 h_3 \succ_4 h_1,\ h_3 \otimes_4 h_4$.

In this context, we know that $\mathcal{C}(\succ) = \{\mu_1, \mu_2, \mu_3\}$ and $\mathcal{C}_{\mathrm{S}}(\succ) = \{\mu_3\}$, where

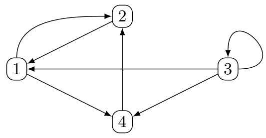
$$\mu_1 = [(1,h_2),(2,h_1),(3,h_3),(4,h_4)], \qquad \mu_2 = [(1,h_2),(2,h_1),(3,h_4),(4,h_3)],$$

$$\mu_3 = [(1,h_4),(2,h_1),(3,h_3),(4,h_2)].$$

The following arguments confirm that μ_3 is the only element in the strong core:

- Since $I_1(\mu_1) = \{2,4\}$, $I_2(\mu_1) = \{1\}$, $I_3(\mu_1) = \{1,3,4\}$, and $I_4(\mu_1) = \{2,3,4\}$,
$$S_{\mu_1} = \{(4,2,1),(3,1,4),(3,4)\}.$$
- Since $I_1(\mu_2) = \{2,4\}$, $I_2(\mu_2) = \{1\}$, $I_3(\mu_2) = \{1,4\}$, and $I_4(\mu_2) = \{2,3,4\}$,
$$S_{\mu_2} = \{(3,1,4),(4,2,1)\}.$$
- Since $I_1(\mu_3) = \{2,4\}$, $I_2(\mu_3) = \{1\}$, $I_3(\mu_3) = \{1,3,4\}$, and $I_4(\mu_3) = \{2\}$, agent 3 is the only one that could strictly improve her situation, by receiving h_1 or h_4.

[19] A *cycle* is an ordered set $\{i_1, i_2, \ldots, i_k\} \subseteq I$ such that $i_k = i_1$ and, for each $s \in \{1,\ldots,k-1\}$, i_s announces i_{s+1}.



However, when every agent i announces the members of $I_i(\mu_3)$ (see figure above), no cycles including agents $\{1,3\}$ or $\{3,4\}$ are formed. Thus, S_{μ_3} is an empty set. □

Example B2. To illustrate the application of our methodology to the housing markets analyzed in Example 4, consider the scenario in which agents' preferences are

$$\succ_1\colon\ h_3 \otimes_1 h_1,\ h_2 \otimes_1 h_1,\ h_2 \otimes_1 h_3; \qquad \succ_2\colon\ h_1 \succ_2 h_2,\ h_3 \otimes_2 h_2,\ h_1 \otimes_2 h_3; \qquad \succ_3\colon\ h_1 \succ_3 h_3,\ h_2 \otimes_3 h_3,\ h_1 \otimes_3 h_2.$$

Applying the TTC algorithm to the completions of $\succ = (\succ_1, \succ_2, \succ_3)$, Theorem 1 ensures that the core of this housing market is given by $\mathcal{C}(\succ) = \{\mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \mu_6\}$, where

$$\begin{aligned}
&\mu_1 = [(1,h_1),(2,h_2),(3,h_3)], && \mu_2 = [(1,h_1),(2,h_3),(3,h_2)], && \mu_3 = [(1,h_2),(2,h_1),(3,h_3)],\\
&\mu_4 = [(1,h_2),(2,h_3),(3,h_1)], && \mu_5 = [(1,h_3),(2,h_1),(3,h_2)], && \mu_6 = [(1,h_3),(2,h_2),(3,h_1)].
\end{aligned}$$

To see that μ_1, μ_3 and μ_6 do not belong to $\mathcal{C}_{\mathrm{S}}(\succ)$, notice that:

- Since $I_1(\mu_1) = I_2(\mu_1) = I_3(\mu_1) = \{1,2,3\}$, we have that $S_{\mu_1} = \{(2,1),(2,1,3),(3,1),(3,1,2)\}$.
- Since $I_1(\mu_3) = I_3(\mu_3) = \{1,2,3\}$, $I_2(\mu_3) = \{1,3\}$, it follows that $S_{\mu_3} = \{(3,1)\}$.
- Since $I_1(\mu_6) = I_2(\mu_6) = \{1,2,3\}$, $I_3(\mu_6) = \{1,2\}$, we have that $S_{\mu_6} = \{(2,1)\}$.

Since $\overline{H}_1 = \{h_1,h_2,h_3\}$, $\overline{H}_2 = \{h_1,h_3\}$, and $\overline{H}_3 = \{h_1,h_2\}$, it follows that $\mathcal{C}_{\mathrm{S}}(\succ) = \{\mu_2,\mu_4,\mu_5\}$. Indeed, in any of these allocations each agent receives one of her best alternatives. □

Analogous arguments to those used in Examples B1 and B2 can be applied to compute the strong core of the other Shapley-Scarf housing markets analyzed in Example 4.

Appendix C – Proof of the claims made in Section 6

Proposition C1. For any preference profile $\succ \in \mathcal{P}^n$ we have that

$$\bigcap_{\widehat{\succ}\,\in\,\mathrm{SC}(\succ)} \mathcal{W}^{\mathrm{IR}}(\widehat{\succ}) \ \subseteq\ \mathcal{E}^{\mathrm{IR}}(\succ) \ \subseteq\ \mathcal{W}^{\mathrm{IR}}(\succ) \ =\ \bigcup_{\widehat{\succ}\,\in\,\mathrm{SC}(\succ)} \mathcal{W}^{\mathrm{IR}}(\widehat{\succ}).$$

Proof. The definitions of weak efficiency and efficiency imply that $\mathcal{E}^{\mathrm{IR}}(\succ) \subseteq \mathcal{W}^{\mathrm{IR}}(\succ)$ for all $\succ \in \mathcal{P}^n$. Moreover, if $\mu \notin \mathcal{W}^{\mathrm{IR}}(\succ)$, then either μ is weakly inefficient or there exists $i \in I_1$ such that $h_i \succ_i \mu(i)$. Hence, for all $\widehat{\succ} \in \mathrm{Co}(\succ)$ we have that $\mu \notin \mathcal{W}^{\mathrm{IR}}(\widehat{\succ})$. Since Proposition 1 ensures that $\mathrm{SC}(\succ) = \mathrm{Co}(\succ)$, it follows that $\bigcup_{\widehat{\succ}\in\mathrm{SC}(\succ)} \mathcal{W}^{\mathrm{IR}}(\widehat{\succ}) \subseteq \mathcal{W}^{\mathrm{IR}}(\succ)$. Analogously, if $\mu \notin \mathcal{E}^{\mathrm{IR}}(\succ)$, then either μ is inefficient or there exists $i \in I_1$ such that $h_i \succ_i \mu(i)$. This implies that there exists $\widehat{\succ} \in \mathrm{Co}(\succ)$ such that $\mu \notin \mathcal{W}^{\mathrm{IR}}(\widehat{\succ})$, which guarantees that $\bigcap_{\widehat{\succ}\in\mathrm{SC}(\succ)} \mathcal{W}^{\mathrm{IR}}(\widehat{\succ}) \subseteq \mathcal{E}^{\mathrm{IR}}(\succ)$.

It remains to prove that $\mathcal{W}^{\mathrm{IR}}(\succ) \subseteq \bigcup_{\widehat{\succ}\in\mathrm{Co}(\succ)} \mathcal{W}^{\mathrm{IR}}(\widehat{\succ})$. Given $\mu \in \mathcal{W}^{\mathrm{IR}}(\succ)$ and $\widehat{\succ} \in \mathrm{Co}(\succ)$, if μ is weakly inefficient under $\widehat{\succ}$, then for any $\sigma \in \mathcal{A}$ that dominates μ under $\widehat{\succ}$ there exists an agent i such that $\mu(i) \otimes_i \sigma(i)$.[20] Let $H_\mu(i)$ be the collection of houses that agent i can obtain in the allocations that dominate μ when preferences are given by some completion of $\succ$:

$$H_\mu(i) = \left\{h \in H \ : \exists \widehat{\succ} \in \mathrm{Co}(\succ),\ \exists \sigma \in \mathcal{A},\ \sigma \text{ dominates}\, \mu \ \text{ and } \ h = \sigma(i)\right\}.$$

[20] By contradiction, suppose that $\sigma \in \mathcal{A}$ dominates μ under $\widehat{\succ}$ and that every agent i can compare $\mu(i)$ with $\sigma(i)$ under $\succ_i$. Let $I' \subseteq I$ be the non-empty set of agents for which $\sigma(i)\,\widehat{\succ}_i\,\mu(i)$. Since $\widehat{\succ} \in \mathrm{Co}(\succ)$ and every agent i knows how to compare $\mu(i)$ with $\sigma(i)$ under $\succ_i$, it must be the case that $\sigma(i) \succ_i \mu(i)$ for all $i \in I'$. Therefore, μ is dominated by σ under $\succ$, which contradicts the fact that $\mu \in \mathcal{W}^{\mathrm{IR}}(\succ)$.

For each $i \in I_1$, let $Q_\mu(i) = \{h \in H_\mu(i) \cup \{h_i\} : h \otimes_i \mu(i)\}$ be the houses in $H_\mu(i) \cup \{h_i\}$ that i does not know how to compare with $\mu(i)$. Analogously, of each $i \in I_2$, define $Q_\mu(i) = \{h \in H_\mu(i) : h \otimes_i \mu(i)\}$.

Let $\widehat{\succ}^* \in \mathrm{Co}(\succ)$ be the preference profile such that $\mu(i) \,\widehat{\succ}_i^*\, h$ for all $i \in I$ and $h \in Q_\mu(i)$.[21]

The following arguments ensure that $\mu \in \mathcal{W}^{\mathrm{IR}}(\widehat{\succ}^*)$:

- If μ is weakly inefficient under $\widehat{\succ}^*$, then there is $\sigma \in \mathcal{A}$ that dominates it. Hence, for some $i \in I$ we have that $\sigma(i) \in Q_\mu(i)$, which implies that $\mu(i) \,\widehat{\succ}_i^*\, \sigma(i)$. This contradicts the fact that μ is dominated by σ.
- If μ is not individually rational under $\widehat{\succ}^*$, then there is an agent $i \in I_1$ such that $h_i \,\widehat{\succ}_i^*\, \mu(i)$. Since $\widehat{\succ}^* \in \mathrm{Co}(\succ)$ and $\mu \in \mathcal{W}^{\mathrm{IR}}(\succ)$, it follows that $h_i \in Q_{\mu(i)}$ and, therefore, $\mu(i) \,\widehat{\succ}_i^*\, h_i$. A contradiction.

We conclude that for any $\mu \in \mathcal{W}^{\mathrm{IR}}(\succ)$ there exists $\widehat{\succ}^* \in \mathrm{Co}(\succ)$ such that $\mu \in \mathcal{W}^{\mathrm{IR}}(\widehat{\succ}^*)$. □

Proposition C2. *Given an ordering f of agents and $g_1, \ldots, g_n \in \mathcal{G}$, the mechanism $\Phi : \mathcal{P}^n \to \mathcal{A}$ defined by $\Phi(\succ_1, \ldots, \succ_n) = \varphi^f(g_1(\succ_1), \ldots, g_n(\succ_n))$ is weakly efficient, individually rational, and group strategy-proof.*

Furthermore, the mechanism $\Psi : \mathcal{R}^n \to \mathcal{A}$ such that $\Psi(\succ_1, \ldots, \succ_n) = \mathrm{TC}(\tau(\succ_1), \ldots, \tau(\succ_n))$ is efficient, individually rational, and weakly group strategy-proof.

Proof. Since $\varphi^f(\succ) \subseteq \mathcal{W}^{\mathrm{IR}}(\succ)$ for every $\succ \in \mathcal{Q}^n$ (see Propositions 1 and 2 in Abdulkadiroğlu and Sönmez (1999)), it follows from Proposition C1 that Φ is weakly efficient and individually rational. Moreover, as φ^f is group strategy-proof (see Pápai (2000)), to show that Φ is group strategy-proof in $\mathcal{P}^n$ it is sufficient to apply analogous arguments to those made in the proof of Theorem 1.

The mechanism TC can be applied to housing allocation problems where tenants and newcomers coexist (see Section 3 in Jaramillo and Manjunath (2012)). Moreover, TC is Pareto efficient, individually rational, and weakly group strategy-proof in the preference domain $\widetilde{\mathcal{R}}^n$ (see Proposition 3 in Jaramillo and Manjunath (2012), Propositions 2 and 3 in Ahmad (2021)). Therefore, since $\mathcal{E}^{\mathrm{IR}}(\succ)$ coincides with the set of Pareto efficient and individually rational allocations of the housing allocation problem with existing tenants and weak preferences $[I_1, I_2, H, (\tau(\succ_i))_{i \in I_1 \cup I_2}]$, analogous arguments to those applied in the proof of Theorem 2 guarantee that Ψ is efficient, individually rational, and weakly group strategy-proof in $\mathcal{R}^n$. □

Proposition C3. *For any preference profile $\succ \in \mathcal{P}^n$ we have that*

$$\bigcap_{\widehat{\succ} \in \mathrm{SC}(\succ)} \bigcup_{f \in \mathcal{F}} \mathrm{SD}^f(\widehat{\succ}) \subseteq \mathcal{E}_{\mathrm{S}}(\succ) \subseteq \mathcal{W}(\succ) = \bigcup_{\widehat{\succ} \in \mathrm{SC}(\succ)} \bigcup_{f \in \mathcal{F}} \mathrm{SD}^f(\widehat{\succ}),$$

where $\mathcal{F}$ is the set of orderings of agents.

Proof. For any profile of preferences $\succ \in \mathcal{Q}^n$, the set of Pareto efficient allocations under $\succ$ coincides with $\bigcup_{f \in \mathcal{F}} \mathrm{SD}^f(\succ)$ (see Theorems 1 and 2 in Svensson (1994)). Therefore, the result follows from identical arguments to those made in the proof of Proposition C1 (ignoring the references to *individual rationality*). □

Proposition C4. *Given an ordering f of agents and $g_1, \ldots, g_n \in \mathcal{G}$, the mechanism $\Phi : \mathcal{P}^n \to \mathcal{A}$ defined by $\Phi(\succ_1, \ldots, \succ_n) = \mathrm{SD}^f(g_1(\succ_1), \ldots, g_n(\succ_n))$ is weakly efficient and group strategy-proof. In addition, the mechanism $\Psi : \mathcal{R}^n \to \mathcal{A}$ such that $\Psi(\succ_1, \ldots, \succ_n) = \mathrm{SD}^f(\tau(\succ_1), \ldots, \tau(\succ_n))$ is efficient and strategy-proof.*

Proof. It follows from Proposition C3 that Φ is weakly efficient. Since SD^f is group strategy-proof in the preference domain $\mathcal{Q}^n$ (see Pápai (2000)), to ensure that the mechanism Φ satisfies this property in $\mathcal{P}^n$ it is sufficient to apply analogous arguments to those made in the proof of Theorem 1.

We know that SD^f is strategy-proof in $\widetilde{\mathcal{R}}^n$ (see Proposition 1 in Svensson (1994)). Also, for each preference profile $\succ = (\succ_i)_{i \in I} \in \mathcal{R}^n$, the set $\mathcal{E}(\succ)$ coincides with the Pareto efficient allocations of the housing allocation problem $[I, H, (\tau(\succ_i))_{i \in I}]$. Therefore, as a consequence of analogous arguments to those applied in the proof of Theorem 2, we conclude that the mechanism Ψ is efficient and strategy-proof in $\mathcal{R}^n$. □

[21] A preference profile with these characteristics always exists (see the proof of Claim A2).