SERIE DOCUMENTOS DE TRABAJO

FACULTAD ECONOMÍA Y NEGOCIOS UNIVERSIDAD DE CHILE

DEPARTAMENTO DE ECONOMÍA

SDT 318

THE SECOND WELFARE THEOREM WITH PUBLIC GOODS IN GENERAL ECONOMIES

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Santiago, julio 2010

The Second Welfare Theorem with public goods in general economies *

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April 22, 2010

Abstract

In this paper we prove a general version of the Second Welfare Theorem for a non-convex and non-transitive economy, with public goods and other externalities in consumption. For this purpose we use the sub-gradient to the distance function (normal cone) to define the pricing rule in this general context.

Keywords: Non-convex separation, Second Welfare Theorem, public goods, externalities.

Subject JEL classification: D11, D61.

1 Introduction

In a convex economic setting, i.e. when the set of preferred elements and the production sets are convex, to our mind one of the first general version of the Second Welfare Theorem (from now on SWT) was proven by Arrow and Debreu (see [1], [7] among others). To demonstrate this result, they assumed general hypotheses on the economy and employed the well known *convex separation property* to obtain a decentralizing vector price that supports a Pareto optimum allocation.

However, it wasn't until the seventies that Guesnerie ([8]) obtained the first general version of the SWT for non-convex economies. For that, the author employed the *Dubovickii-Miljutin's tangent cone* to define the *pricing rule* that allows him to define the corresponding competitive equilibrium concept[1].

*This work was partially supported by FONDAP-Optimización and ICSI, Instituto Sistemas Complejos en Ingeniería, Chile.

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[1]In order to present the SWT in a non-convex setting, the Walras equilibrium concept is replaced by a more general concept based on the employment of the so-called *pricing rule*: for a non-convex economic framework, it does not make sense to assume that agents maximize profit

From Guesnerie's seminal paper, several authors contributed developing an even more general version of the SWT, either considering weaker hypotheses on the fundamentals of the economy and/or employing more general mathematical tools to set the pricing rule in order to present the corresponding results. For instance, in Bonnisseau and Cornet ([4]), Khan and Vohra ([12]) and Yun ([20]), Clarke's normal cone is employed to define the pricing rule (see Clarke ([6])), whereas in Ioffe ([10]), Khan ([13]) and Mordukhovich ([14]) among others, the authors use the normal cone introduced separately by Ioffe and Modukhovich (see Ioffe ([9]) and Mordukhovich ([15])), which allows them to obtain decentralizing prices for a more general economic setting than previously mentioned[2]. Complementarily, see Mordukhovich ([16]), Sec. 8, for a comprehensive discussion on this topic.

In this paper, we will employ the *normal cone* to both preferences and production sets to define our pricing rule (similarly to Khan and Mordukhovich, op. cit.). The main result of this work is Theorem 3.1, which to our mind improves the Khan and Vohra's Theorem 2 in [12] in three aspects. First we assume a global condition over the economy that, as a particular case holds true under the assumptions on preferences and/or production sets they assume. Second, we prove the SWT for a strong Pareto allocation instead of for the weak notion they employ. Finally, as mentioned, we use normal cone instead of Clarke's normal cone to define the pricing rule, which permit us to obtain sharper results in terms of the geometrical conditions we need to assume over preferred and productions sets in order to obtain the desired result.

Since in general the normal cone to a sum of sets at the sum of their elements is not necessarily the intersection of the normal cones to each set at the corresponding points (see Rockafellar and Wets ([18])), contrarily to Khan and Vohra, in our version of the SWT, Theorem 3.1, we are unable to show the existence of a decentralized prices for the public goods sector to each firm individually but for industry, i.e., for the sum of production sets. Conditions that permit the *pass* from industry to individual firms are related with the epi-lipschitzianity and/ot the convexity of the involved sets (see Rockafellar and Wets op.cit.), which are assumed by Khan and Vohra in their contribution. Under the same type of conditions over production sets, we can obtain the same results as they regarding production sector.

This paper is organized as follows. In Section 2 we introduce the model and main concepts, and Section 3 is devoted to demonstrate the main result of the

(firms) or utility (consumers). Instead the equilibrium of an economy is defined according to a rule that corresponds to the first order optimality conditions for an optimization problem that generalizes the usual one that defines both supply and demand for economic agents (consumers and produces), is such a way that under convexity coincide with the standard conditions that determine the Walrasian equilibrium. For general sets, these necessary conditions are defined by means of *normal cones*. Thus, the employment of normal cones appears naturally as an extension of the marginal rate of substitution conditions that usually permit to determine the equilibria allocation of an economy. See Brown ([5]) for a detailed discussion on previous concepts.

[2] The Ioffe-Modukhovich normal cone is called *normal cone* in Rockafellar and Wets ([18]), from which we will adopt the terminology and notation in this paper. What is relevant to our purpose is the fact that the *normal cone* can be calculated for any closed set.

paper, including some direct consequences of it.

2 The model

In this Section we follow Khan and Vohra ([12]) for economic notation and main concepts. Thus, we assume that in the economy there are $\ell \in I\!N \setminus \{0\}$ private consumption goods and $G \in I\!N \setminus \{0\}$ public goods. Public goods are characterized by the fact that their consumption is identical across individuals and they are not subject to congestion (i.e. *pure public goods*). For private and public consumption and/or production we use superscripts π and g respectively.

In the economy there are $m \in I\!N, m \neq 0$, consumers, indexed by $i \in I = \{1, 2, \ldots, m\}$. Each of them is characterized by a consumption set

$$X_i = X_i^{\pi} \times X_i^{G} \subseteq I\!R_+^{\ell+G},$$

and by a preference relation

$$P_i : X_i \times X_{-i} \to X_i,$$

with $X_{-i} = \prod_{k \in I \setminus \{i\}} X_k$. Thus, for $x_{-i} \in X_{-i}$, $P_i(x_i, x_{-i}) \subseteq X_i$ corresponds to the set of *strictly preferred* elements to $x_i \in X_i$ by individual $i \in I$. The closure of this set, $clP_i(x_i, x_{-i})$, denotes the *preferred* elements to x_i by this consumer. Since we are assuming that the preference relation for an individual depends on the consumption of the other agents, we are considering the presence of externalities in consumption besides public goods. Any consumption plan $x_i \in X_i$ can be decomposed in their private and public components, namely $x_i^{\pi} \in I\!R^{\ell}$ and $x_i^{g} \in I\!R^{G}$ respectively (thus, $x_i = (x_i^{\pi}, x_i^{g})$). The projection of $P_i(x_i, x_{-i})$ on $I\!R^{\ell} \times \{0_G\}$ (resp. $\{0_{\ell}\} \times I\!R^{G}$) will be denoted as $P_i^{\pi}(x_i, x_{-i})$ (resp. $P_i^{g}(x_i, x_{-i})$).

In our model we consider the presence of a production sector, characterized by $n \in I\!N$ firms indexed by $j \in J = \{1, 2, ..., n\}$. The set $Y_j \subseteq I\!R^{\ell+G}$ denotes the production set for a firm $j \in J$; $Y_j^{\pi} \subseteq I\!R^{\ell}$ and $Y_j^{g} \subseteq I\!R^{G}$ are defined as in previous paragraph and as for consumers, any production plan $y_j \in Y_j$ can be decomposed in its private and public components, y_j^{π} and y_j^{g} respectively.

Finally, we assume that the total initial endowments of private consumption goods is $\omega^{\pi} \in I\!R_{++}^{\ell}$ and zero for public goods. Let $\omega \equiv (\omega^{\pi}, 0_G) \in I\!R^{\ell} \times I\!R^{G}$ be the vector of total initial endowments of the economy.

An *economy with public goods and other types of externalities* is defined by

$$\mathcal{E}_g = ((X_i)_{i \in I}, (P_i)_{i \in I}, (Y_j)_{j \in J}, \omega).$$

The feasibility of a consumption - production bundle is defined for both private and public components, considering that, by definition, *public goods must be consumed in identical quantities across individuals* (see Khan and Vohra ([12])).

Definition 2.1 *A consumption - production bundle*

$$((x_i),(y_j)) \in I\!R^{m\cdot(\ell+G)} \times I\!R^{n\cdot(\ell+G)}$$

is a feasible allocation for the economy $\mathcal{E}_g$ *if for each* $i \in I$, $j \in J$, *holds that*

(a) $x_i \in X_i$, $y_j \in Y_j$,

(b) $x_i^g = x_{i'}^g$, $i' \in I$,

(c) $\sum\limits_{i\in I} x_i - \sum\limits_{j\in J} y_j = \omega$.

The set of feasible allocations for $\mathcal{E}_g$ *is denoted by* $\mathcal{F}$.

Definition 2.2 *We say that* $\big((x_i^*),(y_j^*)\big) \in \mathcal{F}$ *is a Pareto optimum allocation for the economy* $\mathcal{E}_g$ *if does not exists other feasible allocation* $((\tilde{x}_i),(\tilde{y}_j))$ *such that*

(a) *for every* $i \in I$, $\tilde{x}_i \in clP_i(x_i^*, x_{-i}^*)$,

(b) *for some* $i_0 \in I$, $\tilde{x}_{i_0} \in P_{i_0}(x_{i_0}^*, x_{-i_0}^*)$.

3 The Second Welfare Theorem

The main objective in this Section is to demonstrate a version of the SWT for the economic framework previously described. In order to obtain this result we employ a generalized version of the *convex separation property* demonstrated in Jofré and Rivera ([11]). The key condition there used to establish the separation property is the *Asymptotically Included Condition* (AIC), which for the purpose of this paper can be presented in the following way[3].

Definition 3.1 *We say that* $((x_i^*),(y_j^*)) \in \mathcal{F}$ *satisfies AIC if there exists* $i_0 \in I$, $\varepsilon > 0$, *a sequence* $h_k \to 0_{\ell+G}$ *and* $k_0 \in I\!N$ *such that for all* $k \geq k_0$,

$$-h_k + \sum_{i\in I}[clP_i^* \cap clB(x_i^*,\varepsilon)] - \sum_{j\in J}\Big[Y_j \cap clB(y_j^*,\varepsilon)\Big] \subseteq P_{i_0}^* + \sum_{i\in I\setminus\{i_0\}} clP_i^* - \sum_{j\in J} Y_j,$$

where $P_i^* = P_i(x_i^*, x_{-i}^*)$ *and* $B(x_i^*,\varepsilon)$ *the open ball with center* x_i^* *and radius* $\varepsilon > 0$ *(similarly with* $B(y_j^*,\varepsilon)$*).*

Next proposition provides necessary conditions for AIC.

Proposition 3.1 Necessary conditions for AIC
A point $((x_i^*),(y_j^*)) \in \mathcal{F}$ *satisfies AIC if any of the following holds true*

[3] See Bao and Mordukhovich ([3]) and Mordukhovich ([14, 16]) for the relation among AIC and the *extremal principle*, the extension of this type of condition to infinite dimensional spaces and the relationship with the *net demand qualification conditions* they introduce; see also Rockafellar and Wets ([18]) for an approximate version of this condition.

(a) there exists $i_0 \in I$ such that $x^*_{i_0} \in clP^*_{i_0}$ and the interior of Clarke's tangent cone to $P^*_{i_0}$ at $x^*_{i_0}$, denoted $intT_c(P^*_{i_0}, x^*_{i_0})$, is a non-empty set[4],

(b) there exists $i_0 \in I$ such that $x^*_{i_0} \in clP^*_{i_0}$ and $P^*_{i_0}$ is convex with interior,

(c) there exists $i_0 \in I$ such that for every $x \in clP^*_{i_0}$, $\{x\} + I\!R^{\ell+G}_{++} \subseteq P^*_{i_0}$.

Proof.

(a) From Khan and Vohra ([12]), pag. 229, we know that $y \in intT_c(P^*_{i_0}, x^*_{i_0})$ if and only if there are $\eta > 0$, $\epsilon > 0$ and $\delta > 0$ such that

$$clP^*_{i_0} \cap clB(x^*_{i_0}, \delta) + [0, \eta]clB(y, \epsilon) \subseteq P^*_{i_0}.$$

Thus, if for some i_0, $intT_c(P^*_{i_0}, x^*_{i_0}) \neq \emptyset$ we can readily obtain the result.

(b) From AIC we know that there are $\tilde{x} \in P^*_{i_0}$ and $\delta > 0$ such that $clB(\tilde{x}, \delta) \subseteq P^*_{i_0}$. Given $x^*_{i_0} \in clP^*_{i_0}$, from Rockafellar ([17]), Theorem 6.1, given $\epsilon > 0$ and $0 < \delta_1 < \delta$, holds that for every $\lambda \in [0, 1[$

$$(1 - \lambda)clB(\tilde{x}, \delta_1) + \lambda \left[clP^*_{i_0} \cap clB(x^*_{i_0}, \epsilon)\right] \subseteq P^*_{i_0}.$$

Let $\{\lambda_k\}$ a real sequence such that $\lambda_k \to 1^-$. Given $\epsilon_1 > 0$ and

$$h_k = (1 - \lambda_k) \cdot (\tilde{x} - x^*_{i_0}) \to 0 \in I\!R^\ell$$

define $z \in h_k + clP^*_{i_0} \cap clB(x^*_{i_0}, \epsilon_1)$. From hypothesis, there exists $x' \in clP^*_{i_0} \cap clB(x^*_{i_0}, \epsilon_1)$ such that $z = h_k + x'$, that is,

$$z = (1 - \lambda_k)\tilde{x} - (1 - \lambda_k)x^*_{i_0} + x' = (1 - \lambda_k)\left[\tilde{x} + x' - x^*_{i_0}\right] + \lambda_k x'.$$

Note that for ϵ_1 small enough, $\left(\tilde{x} + x' - x^*_{i_0}\right) \in clB(\tilde{x}, \delta_1)$, and then, given ϵ_1 as before, we conclude that

$$z \in (1 - \lambda_k)clB(\tilde{x}, \delta_1) + \lambda_k \left[clP^*_{i_0} \cap clB(x^*_{i_0}, \epsilon_1)\right] \subseteq P^*_{i_0},$$

i.e., $h_k + clP^*_{i_0} \cap clB(x^*_{i_0}, \epsilon_1) \subseteq P^*_{i_0}$, which ends the proof.

(c) This part is obvious if we note that this condition is equivalent to assume that

$$clP^*_{i_0} + I\!R^{\ell+G}_{+} \subseteq P^*_{i_0},$$

and therefore is valid for $clP^*_{i_0} \cap clB(x^*_{i_0}, \epsilon)$, $\epsilon > 0$. E.O.P

[4]For the Clarke's tangent cone definition, see Clarke ([6]).

In order to establish our main result (Theorem 3.1) we will employ the following assumptions, which are quite standard in the literature.

Assumption C. For each $i \in I$, $X_i = I\!R_+^{\ell+G}$.

Assumption P. For each $j \in J$, Y_j is a closed set.

Assumption D. Public goods are *desirable* for each individual, that is, for $i \in I$ and $z_i \in clP_i(x_i, x_{-i})$, given $h \in I\!R_{++}^G$ holds that

$$z_i + (0_\ell, h) \in P_i(x_i, x_{-i}).$$

Assumption B. For every $i \in I$, $x_i \in clP_i(x_i, x_{-i}) \setminus P_i(x_i, x_{-i})$.

Assumption F. For some $j_0 \in J$, Y_{j_0} satisfies the *free disposal hypothesis*, i.e., $Y_{j_0} - I\!R_+^{\ell+G} \subseteq Y_{j_0}$.

Lemma 3.1 Boundary property
Let $((x_i^*), (y_j^*))$ be a Pareto optimum for economy $\mathcal{E}_g$ that satisfies AIC. If **C**, **D**, **B** *and* **F** *are verified, then*

$$w \in bd\left[\sum_{i\in I} clP_i^* - \sum_{j\in J} Y_j\right].$$

Proof. For $\varepsilon > 0$, let us define

$$\Gamma_\varepsilon = \sum_{i\in I}\left[clP_i^* \cap clB(x_i^*, \varepsilon)\right] - \sum_{j\in J}\left[Y_j \cap clB(y_j^*, \varepsilon)\right].$$

From **F**, for each $\varepsilon > 0$ we have that $int\Gamma_\varepsilon \neq \emptyset$ and, moreover, from **B** we also have that $\omega \in \Gamma_\varepsilon$. Now, if for some $\varepsilon_0 > 0$, $\omega \notin bd\Gamma_{\varepsilon_0}$, then, from previous considerations, follows that $\omega \in int\ \Gamma_{\varepsilon_0}$, i.e., for each sequence $v_k \to 0_{\ell+G}$, there exists $K \in I\!N$ such that $\omega + v_k \in \Gamma_{\varepsilon_0}$, for all $k \geq K$. This last condition along with AIC directly imply that for some $i_0 \in I$

$$\omega \in P_{i_0}^* + \sum_{i\in I\setminus\{i_0\}} clP_i^* - \sum_{j\in J} Y_j,$$

that is, there exists $\bar{x}_{i_0} \in P_{i_0}^*$, $\bar{x}_i \in clP_i^*$, $i \neq i_0$ and $\bar{y}_j \in Y_j$ such that

$$\omega = \sum_{i\in I}\bar{x}_i - \sum_{j\in J}\bar{y}_j.$$

Given $\delta > 0$, for $i \in I$ define $\tilde{x}_i = (\tilde{x}_i^\pi, \tilde{x}_i^g)$, with

$$\tilde{x}_i^\pi = \bar{x}_i^\pi, \quad \tilde{x}_i^g = \left[\max_{s\in I}\{\bar{x}_{is}^g\} + \delta\right]\mathbf{1}_G,$$

where $\mathbf{1}_G = (1,1,\ldots,1) \in I\!R^G$. Note that $\tilde{x}_i^g = \tilde{x}_{i'}^g, i, i' \in I$, and from hypotheses **C** and **D** we have that for each $i \in I$, $\tilde{x}_i \in X_i$ and $\tilde{x}_i \in P_i^*$ respectively. On the other hand, by construction $\sum_{i\in I}[\tilde{x}_i - \bar{x}_i] \in I\!R_+^{\ell+G}$ and then, defining

$$\tilde{y}_j = \bar{y}_j,\ j \in J \setminus \{j_0\},\ \ \tilde{y}_{j_0} = \bar{y}_{j_0} - \sum_{i\in I}[\tilde{x}_i - \bar{x}_i],$$

from **F** follows that for each $j \in J$, $\tilde{y}_j \in Y_j$. Given all the foregoing, it is easy to check that $((\tilde{x}_i),(\tilde{y}_j))$ is a feasible allocation that contradicts the optimality of $((x_i^*),(y_j^*))$, which ends the proof. **E.O.P**

In the remaining part of this work, the *normal cone* to a closed set $A \subseteq I\!R^n$ at $a \in I\!R^n$ is denoted by $N(A,a)$. Following properties of the normal cone will be used in the demonstration of Theorem 3.1 (see Rockafellar and Wets ([18]) for details): (i) for every $\epsilon > 0$, $N(A \cap clB(x,\epsilon), x) = N(A,x)$ (*local property*), (ii) for any couple of closed sets $A, B \subseteq I\!R^n$ and $a, b \in I\!R^n$, $N(A\times B,(a,b)) = N(A,a)\times N(B,b)$ (*product property*), and (iii) for every $\lambda \in I\!R_{++}$, $N(\lambda A, \lambda a) = N(A,a)$ (*homogeneity property*) .

Finally, for a Pareto optimum allocation $((x_i^*),(y_j^*))$ of economy $\mathcal{E}_g$, for $i \in I$ we denote $P_i^{*\pi} = P_i^\pi(x_i^*, x_{-i}^*)$ and $P_i^{*g} = P_i^g(x_i^*, x_{-i}^*)$.

Theorem 3.1 Second Welfare Theorem

Let $((x_i^*),(y_j^*))$ be a Pareto optimum allocation for economy $\mathcal{E}_g$. If **C**, **P**, **D**, **B** *and* **F** *are satisfied, then there are prices $p^\pi \in I\!R^\ell$ and $p_i^g \in I\!R^G$, $i \in I$, not all zero, such that*

$$-(p^\pi, p_i^g) \in N\left(clP_i^*, x_i^*\right) \tag{1}$$

$$p^\pi \in \bigcap_{j\in J} N\left(Y_j^\pi, y_j^{*\pi}\right) \tag{2}$$

$$\sum_{i\in I} p_i^g \in N\left(\sum_{j\in J} Y_j^g, \sum_{j\in J} y_j^{*g}\right) \tag{3}$$

Proof. For $\epsilon > 0$ and $i \in I$, define

$$clP_i^{*\pi}(\epsilon) = clP_i^{*\pi} \cap clB(x_i^{*\pi},\epsilon) \subseteq I\!R^\ell,\ \ clP_i^{*g}(\epsilon) = clP_i^{*g} \cap clB(x_i^{*g},\epsilon) \subseteq I\!R^G.$$

Clearly previous sets are non-empty and closed. Indeed, from **B**,we have that

$$x_i^* = (x_i^{*\pi}, x_i^{*g}) \in clP_i^{*\pi}(\epsilon) \times clP_i^{*g}(\epsilon).$$

For $i \in I = \{1,2,\ldots,m\} \setminus \{1,m\}$, and $\epsilon > 0$ define now

$$A_i(\epsilon) = clP_i^{*\pi}(\epsilon) \times \{0_G\}^{i-1} \times clP_i^{*g}(\epsilon) \times \{0_G\}^{m-i} \subseteq I\!R^\ell \times I\!R^{mG},$$

and

$$A_1(\epsilon) = clP_1^{*\pi}(\epsilon) \times clP_1^{*g}(\epsilon) \times \{0_G\}^{m-1}, \;\; A_m(\epsilon) = clP_m^{*\pi}(\epsilon) \times \{0_G\}^{m-1} \times clP_m^{*g}(\epsilon).$$

From definition we have that for $i, i' \in I$, $x_i^{*g} = x_{i'}^{*g}$. Denote this value by x^{*g} and therefore, from feasibility conditions hold that

$$x^{*g} = \frac{1}{m} \sum_{j \in J} y_j^{*g} \in \frac{1}{m} \sum_{j \in J} Y_j^g \subseteq I\!R^G.$$

For $\epsilon > 0$ and $j \in J = \{1, 2, \ldots, n\}$, define now

$$Y_j^\pi(\epsilon) = Y_j^\pi \cap clB(y_j^{*\pi}, \epsilon), \quad \mathbf{Y}^g = \frac{1}{m} \sum_{j \in J} Y_j^g, \quad \mathbf{Y}^g(\epsilon) = \frac{1}{m} \sum_{j \in J} Y_j^g \cap clB(\mathbf{y}^{*g}, \epsilon),$$

with

$$\mathbf{y}^{*g} = \frac{1}{m} \sum_{j \in J} y_j^{*g}.$$

In order to continue with the demonstration, regarding the number of agents we should consider two cases: $m \geq n$ and $m < n$. For the case $m \geq n$, similarly to the $A_i(\epsilon)$ definition, with the precaution for the cases 1 and m as before, for $m > n$ and $k \in I$ define $B_k(\epsilon)$ as follows:

$$1 \leq k \leq n: \quad B_k(\epsilon) = Y_k^\pi(\epsilon) \times \{0_G\}^{k-1} \times \mathbf{Y}^g(\epsilon) \times \{0_G\}^{m-k} \subseteq I\!R^\ell \times I\!R^{mG}$$

$$n < k \leq m: \quad B_k(\epsilon) = \{0_\ell\} \times \{0_G\}^{k-1} \times \mathbf{Y}^g(\epsilon) \times \{0_G\}^{m-k} \subseteq I\!R^\ell \times I\!R^{mG}.$$

When $m = n$ we omit the the case $n < k \leq m$ above. Finally, for $i, k \in I$ define

$$\mathbf{x}_i^* = (x_i^{*\pi}, 0_G, \cdots, 0_G, x^{*g}, 0_G, \cdots, 0_G) \in A_i(\epsilon),$$
$$\mathbf{y}_k^* = (z_k^*, 0_G, \cdots, 0_G, \mathbf{y}^{*g}, 0_G, \cdots, 0_G) \in B_k(\epsilon),$$

with $z_k^* = y_k^{*\pi}$ if $1 \leq k \leq n$ and 0_ℓ otherwise.

For the case $m < n$, given $1 \leq k \leq n$ and $\epsilon > 0$, with the corresponding precaution as before, define

$$B_k(\epsilon) = Y_k^\pi(\epsilon) \times \{0_G\}^{k-1} \times \mathbf{Y}^g(\epsilon) \times \{0_G\}^{n-k} \subseteq I\!R^\ell \times I\!R^{nG},$$

and $A_k(\epsilon)$ considering the following cases

$$1 \leq k \leq m: \quad A_k(\epsilon) = clP_k^{*\pi}(\epsilon) \times \{0_G\}^{k-1} \times clP_k^{*g}(\epsilon) \times \{0_G\}^{n-k} \subseteq I\!R^\ell \times I\!R^{nG},$$

$$m < k \leq n: \quad A_k(\epsilon) = \{0_\ell\} \times \{0_G\}^{k-1} \times clP_1^{*g}(\epsilon) \times \{0_G\}^{n-k} \subseteq I\!R^\ell \times I\!R^{nG}.$$

Thus, given $1 \leq k \leq n$ and $\epsilon > 0$, define

$$\mathbf{x}_k^* = (z_k^*, 0_G, \cdots, 0_G, x^{*g}, 0_G, \cdots, 0_G) \in A_k(\epsilon),$$
$$\mathbf{y}_k^* = (y_k^{*\pi}, 0_G, \cdots, 0_G, \mathbf{y}^{*g}, 0_G, \cdots, 0_G) \in B_k(\epsilon),$$

with $z_k^* = x_k^{*\pi}$ if $1 \leq k \leq m$ and 0_ℓ otherwise.

Under any of the previous situations regarding the number of agents in the economy, following reasoning we develop for the case $m > n$ conducts to the same conclusions. Given that, it is easy to check that for $\epsilon > 0$

$$\sum_{k \in I} B_k(\epsilon) = \left[\sum_{j \in J} Y_j^\pi(\epsilon)\right] \times [\mathbf{Y}^g(\epsilon)]^m \subseteq I\!R^\ell \times I\!R^{mG},$$

and

$$\sum_{i \in I} \mathbf{x}_i^* - \sum_{k \in I} \mathbf{y}_k^* \in \sum_{i \in I} A_i(\epsilon) - \sum_{k \in I} B_k(\epsilon).$$

Moreover, from Lemma 3.1 we can also assert

$$\sum_{i \in I} \mathbf{x}_i^* - \sum_{k \in I} \mathbf{y}_k^* \in bd\left[\sum_{i \in I} A_i(\epsilon) - \sum_{k \in I} B_k(\epsilon)\right],$$

and then, from the separation property in Jofré and Rivera ([11]) and the *local property* for normal cones (which permit us to avoid the balls in the calculus of normal cones immediately below), there exists a non-null vector price $p = (p^\pi, p_1^g, ..., p_m^g) \in I\!R^\ell \times I\!R^{mG}$, such that

$$-p \in N\left(\sum_{i \in I} clP_i^{*\pi} \times \prod_{i \in I} clP_i^{*g}, \left(\sum_{i \in I} x_i^{*\pi}, x^{*g}, \cdots, x^{*g}\right)\right)$$

and

$$p \in N\left(\sum_{j \in J} Y^{*\pi} \times [\mathbf{Y}^g]^m, \left(\sum_{j \in J} y_j^{*\pi}, \mathbf{y}^{*g}, \cdots, \mathbf{y}^{*g}\right)\right).$$

Separating private and public components in previous relations (*product property*), holds that for each $i \in I$

$$-p^\pi \in N\left(\sum_{i \in I} clP_i^{*\pi}, \sum_{i \in I} x_i^{*\pi}\right), \quad p^\pi \in N\left(\sum_{j \in J} Y_j^{*\pi}, \sum_{j \in J} y_j^{*\pi}\right),$$

$$-p_i^g \in N\left(clP_i^{*g}, x^{*g}\right), \quad p_i^g \in N\left(\mathbf{Y}^g, \mathbf{y}^{*g}\right).$$

On the other hand, considering the *homogeneity property*, we have that

$$N\left(\mathbf{Y}^g, \mathbf{y}^{*g}\right) = N\left(\frac{1}{m}\sum_{j\in J} Y_j^g, \frac{1}{m}\sum_{j\in J} y_j^{*g}\right) = N\left(\sum_{j\in J} Y_j^g, \sum_{j\in J} y_j^{*g}\right),$$

which directly implies that

$$\sum_{i\in I} p_i^g \in N\left(\sum_{j\in J} Y_j^g, \sum_{j\in J} y_j^{*g}\right).$$

Furthermore, considering that

$$\omega^\pi = \sum_{i\in I} x_i^{*\pi} - \sum_{j\in J} y_j^{*\pi} \in bd\left[\sum_{i\in I} clP_i^{*\pi} - \sum_{j\in J} Y_j^\pi\right],$$

from Jofré and Rivera op.cit. we finally conclude that

$$-p^\pi \in \bigcap_{i\in I} N\left(clP_i^{*\pi}, x_i^{*\pi}\right), \quad p^\pi \in \bigcap_{j\in J} N\left(Y_j^{*\pi}, y_j^{*\pi}\right),$$

which ends the proof. **E.O.P**

We remark that statements (1) - (3) in Theorem 3.1 can be considered as the natural extension of the well know Samuelson' condition for the assignment of public goods in production economies (see [19]). However, in our framework we are unable to show that the supporting price for public goods ($p^g = \sum_{i\in I} p_i^g$) satisfies the optimality conditions for each firm but for the *industry*

$$p^g = \sum_{i\in I} p_i^g \in N\left(\sum_{j\in J} Y_j^g, \sum_{j\in J} y_j^{*g}\right).$$

From the mathematical point of view, this *incompatibility* arises from the fact that, in general, the normal cone to the sum of sets at a sum of points is not necessarily included in the intersection of the corresponding normal cones (see Rockafellar and Wets, op.cit.). In order to obtain a *compatibility* between firms and industry as mentioned, we need extra conditions over the productions sets, which are usually assumed in the literature. Examples of these conditions are:

(a) the convexity of the public production component of any firm (Y_j^g) (see [18], Pag. 230, for details),

(b) all sets Y_j satisfy the free disposal hypothesis,

(c) all sets Y_j are epi-lipschitzian sets[5],

[5] We recall that a set $Y \subseteq I\!R^\ell$ is epi-lipschitzian at a point $y \in Y$ if there exists $d \in I\!R^\ell \setminus \{0_\ell\}$ and a couple of open neighborhoods N_y and N_d of $y \in Y$ and $d \in I\!R^\ell$ respectively, and $\lambda > 0$ such that for each $y' \in Y \cap N_y$ and $t \in (0, \lambda)$, $y' + tN_d \subseteq Y$. See [18] for more details on this concept.

(d) the Clarke's tangent cone to the sets Y_j at the Pareto optimum allocation has nonempty interior (see [12]).

Thus, under any of previous conditions (a) - (d), combining statements (2) and (3) in Theorem 3.1, we can readily show that $p^* = (p^\pi, \sum_{i\in I} p_i^g)$ satisfies

$$p^* \in \bigcap_{j\in J} N(Y_j, y_j^*).$$

It is worth to mention that condition (b) above is used by Khan and Vohra ([12]) to demonstrate that the interior of the tangent cone to the sum of production sets is non-empty, which permits them to conclude that the supporting price belongs to the normal cone to each production set instead of the aggregate production sector (industry) as our general result.

Finally, in spite of the above mentioned for the general case, from the sum formula in Rockafellar and Wets ([18]), Ch. 6, we can present an *approximated version* of the SWT as follows. Thus, from this formula we already know that there are $\overline{y}_j^g \in Y_j^g, j \in J$, such that $\sum_{j\in J} \overline{y}_j^g = \sum_{j\in J} y_j^{*g}$ and

$$p^g \in \bigcap_{j\in J} \in N\left(Y_j^g, \overline{y}_j^g\right).$$

Given that, Theorem 3.1 can be re-writen equivalently in the following way.

Theorem 3.2 *Let $((x_i^*), (y_j^*))$ be a Pareto optimum for economy $\mathcal{E}_g$. If* **C**, **P**, **D**, **B** *and* **F** *are satisfied, then there are prices $p^\pi \in I\!R^\ell$, $p_i^g \in I\!R^G$, $i \in I$, not all zero, and production plans $\overline{y}_j^g \in Y_j^g, j \in J$, such that for each $i \in I$*

$$\sum_{j\in J} y_j^{*g} = \sum_{j\in J} \overline{y}_j^g$$
$$-(p^\pi, p_i^g) \in N\left(clP_i^*, x_i^*\right)$$
$$(p^\pi, \sum_{i\in I} p_i^g) \in \bigcap_{j\in J} N\left(Y_j, (y_j^{*\pi}, \overline{y}_j^g)\right).$$

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