A NON-CONVEX EQUILIBRIUM MODEL WHEN PRODUCERS HAVE MANY PRODUCTION ALTERNATIVES

JORGE RIVERA CAYUPI

Abstract

This paper is on general equilibrium theory, in finite dimensional spaces, where is considered explicitly the existence of exogenous parameters that may affect productivity of firms. Those parameters could be associated with external restriction or possibilities to produce as, for instance, size of the firm or technical options to adopt. In the model will be assumed that for each firm these parameters that defines technology of production are a decision variable for firms, which generalizes the standard model where technology is fixed a priory. The main result of the paper is the existence of equilibrium theorem under general assumptions over the economy, in particular the presence of non-convexities in production.

Resumen

Este trabajo es sobre teoría del equilibrio general en espacios de dimensión finita, donde se considera explícitamente la existencia de un parámetro exógeno que puede afectar la productividad de las firmas. Este parámetro podría ser asociado con restricciones externas o posibilidades de producción como, por ejemplo, el tamaño de la firma u opciones técnicas a adoptar por su parte. En el modelo se asumirá que para cada firma estos parámetros que definen la tecnología son una variable de decisión para la firma, lo que generaliza el modelo estándar donde la tecnología es fija y dada a priori. El principal resultado de este trabajo es un teorema de existencia de equilibrio bajo hipótesis generales sobre la economía, en particular, bajo supuestos de no-convexidad en la producción.

Keywords: Non-convexities in production, general equilibrium, technical options.

JEL Codes: D21, D41, D51.

☐ Departamento de Economía, Universidad de Chile. Financiado por Proyecto Fondecyt 1000766 del 2000.

1. Introduction

In the standard Arrow-Debreu model on general equilibrium theory (Arrow and Debreu, 1964; Debreu, 1959) and in subsequent generalizations (Bonnisseau and Cornet, 1988b.; Quinzii, 1992, see references therein.), each firm $j \in J := \{1.., n\}$ is characterized by a set $Y_j \subset \mathbb{R}^\ell$, where ℓ denotes the number of goods in economy. This set represent all of the feasible input-output combinations of goods, that is, the technology of production for them.

In the convex case[1], given a vector price $p \in \mathbb{R}^\ell$, the economical problem for every firm $j \in J$ consists in to find an optimal production plan $y_j \in Y_j$ that maximizes $p \cdot y_j$ over Y_j. In aforementioned models, the only decision variable for each firm is the optimal production plan and thus technology itself is not a part of the decision for firms. Previous fact can be considered as an economical short-run restriction of the model. However, in spite of all foregoing, there are many cases in economy where some firms, before participate in the market, decide on the technology they will use to produce. In such case, once technology is adopted among certain possibilities, could be reasonable to assume it as permanent, which would correspond to framework of the standard model. As an example of previous situation, the election of the spatial location for production plants could be a very relevant decision problem for firms, which in several cases must be solved previously to any other decision of production or commercial strategies. Another example are firms that previous to produce must decide over several alternatives to set up the process itself, considering the existence of a wide variety of alternatives to do it. This is the case, for instance, in telecommunication sector, where firm must decide on the type and the size of telephone plants, type and size of transmission lines (optical fiber, cooper, air, etc.).

From previous examples, a more realistic production model in general equilibrium theory would have to consider this *two stage reality*, where the first stage consists on a technological decision and secondly on optimal production using it.

In the model developed in this paper will be assumed explicitly that firms must decide on production technology among certain set of possibilities. If this set consist in just one point, this new model corresponds to the standard one. The rest of the assumption are similar to the standard non-convex model of general equilibrium theory (see Brown, 1991 for more details on non-convexities in general equilibrium models).

2. The Model and Main Definitions

In what follows we denote by $\ell, m, n \in \mathbb{N} \setminus \{0\}$ the finite number of goods, consumers and producers respectively. The consumption set for

[1] That is, when Y_j is a convex set. For more general models (non-convex) the max profit condition is replaced by a more general one that, roughly speaking, consist in to find a production plan which satisfies certain optimality condition according to a *pricing rule* properly defined in the model (see more details in Bonnisseau and Cornet, 1988b.).

individual $i \in I = \{1,\ldots, m\}$ will be $X_i = \mathbb{R}^\ell_+$ whereas tastes of them will be described by an utility function $u_i : X_i \to \mathbb{R}$. Let $w_i \in \mathbb{R}^\ell$ be the initial endowment for each consumer $i \in I$ and $w = \sum_{i \in I} w_i$ be the total initial endowment for the economy.

From the producer point of view, as we mentioned previously, in the standard Arrow-Debreu's model and subsequent generalizations, each firm $j \in J = \{1, 2,\ldots, n\}$ it is described by a set $Y_j \subseteq \mathbb{R}^\ell$, which summarizes its production technology. As we mentioned, this technology is assumed fixed and given a priory for every firm in the economy.

As we mentioned, in the model developed in this paper will be considered explicitly the existence of external parameters in the economy that may affect the production of goods, which finally define a set of attainable technologies for every firm. These parameters, among optimal production plans, will be assumed as decision variables for firms. As an example of this situation we may suppose that these parameters could define different exogenous variable that may affect productivity or capability of the firms.

To formalize the idea, let $E \in \mathbb{N}$ be the number of parameters we are taking into account as external variables and let $V \subseteq \mathbb{R}^E$ be a non-empty set of the possible values of these parameters.

Definition 2.1.

The *set of admissible technologies* for a firm $j \in J$ will be described by the images of a set valued mapping[2]

$$Y_j : V \rightrightarrows \mathbb{R}^\ell$$

which, by abuse of language, will be called the *production set* for a firm $j \in J$.

Example 2.1.

Let us suppose that for some firm the set of feasible technologies is defined by a Cobb-Douglas type of production function, let say $f(y_1, y_2) = y_1^\alpha y_2^\beta$ (two inputs, one output), where $\alpha, \beta \in]0,1]$. In such case we have that $E = 2$ and $V =]0,1] \times]0,1]$. In this case, given $(\alpha_0, \beta_0) \in V$, the set valued map Y_j evaluated at that point is the following set:

$$Y_j(\alpha_0, \beta_0) = \{(y_1, y_2, y_3) \mid y_3 \le y_1^{\alpha_0} y_2^{\beta_0}\}.$$

2 We recall that a set valued mapping Φ from A to B is a map such that for every $a \in A$, $\Phi(a) \subseteq B$. As a particular case, an usual *function* corresponds to the case when $\Phi(a)$ is a singleton, that is a set with only one element.

Remark 2.1.

From the producer point of view, one argue that this model is just a particular case of the standard one because variable $v \in V$ could be assumed as another good (input) in the economy. Thus, production set would be

$$Y(V)_j = \bigcup_{v \in V} (v, Y_j(v)) \subset \mathbb{R}^{\ell + E}.$$

However this argumentation is not valid because, as we shall see later, this *new good* $v \in V$ will not have associated prices, and then will not participate in the equilibrium (exchange) as an standard good. Certainly at the equilibrium these exogenous variables must be closely related with prices.

Previous fact does not means that technologies are free: in fact, wherever a firm adopts a production set among feasible points given by the set valued mapping Y_j, then are implicitly defined cost and revenues mappings for each of them.

Finally, note that the model where technology is given a priori is a particular case of this model and corresponds to consider $Y_j(\cdot)$ constant.

Finally, given the simplex in $\mathbb{R}^\ell$

$$S := \{(p_j) \in \mathbb{R}^\ell_+ | \sum_{j \in J} p_j = 1\}$$

the *wealth* of i th consumer will be defined as the map

$$r_i : S \times \mathbb{R}^{\ell n} \to \mathbb{R}$$

which will associate prices and productions plans with income for each individual[3].

3. Equilibrium in this model

In what follows we are going to define an equilibrium notion for the model. The main idea of this paper, as has been mentioned, is to incorporate explicitly the election of the *external technological parameters* as another decision variable for each firm.

In order to illustrate the concept that will defined, given $j \in J$ let us suppose that for any $v \in V$, the set $Y_j(v)$ is convex and let us suppose it is given a vector price p. In order to obtain the *optimal technology* and the *optimal production plan*, each firm $j \in J$ must solve the following optimization problem:

---

3 A particular case of this map is $r_i(p,(y_j)) = p \cdot w_i + \sum_{j \in J} \theta_{ij} p \cdot y_j$ where $\theta_{ij} \geq 0, i \in I, j \in J$ and $\sum_{i \in I} \theta_{ij} = 1$: shares of the individual i in firm j. This model represent a private ownership economy. See Bonnisseau and Cornet (1988b.), and Debreu (1959) for more details.

$$\max_{y_j, v}\; p \quad y_j$$
$$y_j \quad Y_j(v)$$
$$v \quad V.$$

that is, choose a technology among the admissible ones and an optimal production plan given this technology. Note that the in the problem the production-technology election is simultaneous.

Following Bonnisseau and Cornet (1988b.), let $\;_j : Y_j(V) \quad S \quad V$ and $: \quad _{j \; J} Y_j(V) \quad (S \quad V)^n$ be the following set valued mappings

$$_j(y) := \{(p, v) \,|\, p \quad y \quad p \quad y \quad y \quad Y_j(v)\, , y \quad Y_j(v\;), v \quad V\}$$

$$((y_j)) = \;_{j \; J} \quad _j(y_j).$$

Therefore, if we denote by v_j and y_j, $j \quad J$, the solution of previous optimization problem, it must be valid that

$$(p \;, v \;) \quad ((y_j))$$

and

$$y_j \quad Y_j(v_j)\; j \quad J$$

Note that under suitable condition over sets one may argue that the optimal production plan of the last optimization problem must lie on the boundary of the production sets. This is the case, for instance, if we assume that for every $v \quad V$, set $Y_j(v)$ satisfies *free disposal hypothesis*, that is,

$$Y_j(v) \quad \mathbb{R}^{\ell}_{+} \quad Y_j(v) \quad v \quad V$$

The proof is immediate and assumptions that ensure this property will be made in this paper and thus pricing rules will be defined considering that take it values on the boundary of the sets instead of whole set.

Finally, previous approach can be readily extended to consider more general cases than convex production sets.

Definition 3.1.

A *pricing rule* will be any set valued map

$$: \quad _j \; Gr[bd(Y_j)] \mapsto (S \quad V)^n$$

where $Gr[bd(Y_j)] := \{(v_j, y_j) \,|\, v_j \quad V, y_j \quad bd[Y_j(v_j)]\}$ is the graph of the *boundary* of the *production* set valued map.

Remark 3.1.

There are several ways to define pricing rules in economy. In particular we can consider extensions of the average pricing rule, the free loss pricing rule, the marginal cost pricing rule, and so on. See Bonnisseau and Cornet (1988a.), and Brown (1991) for more details.

Definition 3.2.

An economy with several alternatives in the production sector is defined as

$$E_V = ((X_i),(u_i),(r_i),(w_i),(Y_j), \quad ,V).$$

Definition 3.3.

A point $(v \ , p \ ,(x_i),(y_j)) \quad V^n \quad S \quad \mathbb{R}^{\ell m} \quad \mathbb{R}^{\ell n}$ is an *equilibrium point* for the economy E_V if

a.– For all i, $x_i \quad X_i$ maximize $u_i(\ )$ on the budget set

$$\{x_i \quad X_i | \ p \quad x_i \quad r_i(p \ ,(y_j))\}.$$

b.– For all $j = 1,...,n$, $y_j \quad bdY_j(v_j)$ and

$$((v_j, p \ )) := (v_1, p \ , v_2, p \ ,..., v_n, p \ ) \qquad ((v_j, \ y_j))$$

c.– $\quad {}_i \, x_i \qquad {}_j \, y_j = w$.

Remark 3.2.

In the particular case of a separable pricing rule, that is when exist n set valued mappings $\ _j : Gr \ \ bd\,Y_j \ \mapsto S \quad V$, $j \quad J$, such that

$$= \ {}_{j \ J} \quad {}_j$$

condition $b)$ can be replaced by: $b\ ) \ (v_j, p \ ) \quad {}_j(v_j, \ y_j)$, for all $j \quad J$.

Definition 3.4.

Given a pricing rule $\quad$, we define the *set of production equilibrium* as

$$PE = \{(p,(v_j, y_j)) \quad S \qquad {}_j \, Gr[bd\,Y_j] \,|\, ((v_j, \ p)) \qquad ((v_j, \ y_j))\}$$

3.1. *Some hypotheses and a theorem*

In what follows, we will give some conditions over the economy in order to ensure the existence of equilibrium. These conditions will be assumed on the consumption and production sectors and, of course, over the set V. To do that, we follow the model developed in Bonnisseau and Cornet (1988b.) but considering the existence of this new parameter that affects the behavior of the firms.

Hypotheses we are going to impose can by divided into four groups as follows.

(i) Hypotheses on the consumption sector.

(C) For each $i \quad \{1,\ldots, m\}$, we assume that

- $X_i = \mathbb{R}^{\ell}_{+}$
- u_i is continuous, quasiconcave, locally nonsatiated
- $w_i \gg 0$
- r_i is continuous and homogeneous of degree 1 and

$$\sum_{i=1}^{m} r_i(p,(y_j)) = p \quad \sum_{j=1}^{n} y_j + w \quad .$$

(ii) Hypotheses on the production sector.

(P) For each $j \quad \{1,\ldots, n\}$ we shall assume that

- Y_j is l.s.c with closed graph
- for each $v_j \quad V$, $Y_j(v_j) \bigcap \mathbb{R}^{\ell}_{+} = \{0\}$.
- for each $v_j \quad V$, $Y_j(v_j) \quad \mathbb{R}^{\ell}_{+} = Y_j(v_j)$.

(PR) We shall assume that   is u.s.c. with nonempty convex compact values.

(NL) For all $((v_j, y_j)) \quad {}_j Gr[bd\,Y_j]$ and for all $((v_j, p_j)) \quad ((v_j, y_j))$ we have that $p_k \quad y_k \quad 0$, $k \quad \{1,\ldots, n\}$.

(iii) Hypotheses on the global economy.

(B) For every $v = (v_j) \quad V^n$, the set

$$A(v) = \{(y_j) \quad {}_j Y_j(v_j) | \; 0 \quad {}_j y_j + w\}$$

is uniformly bounded with respect to $v \quad V$, which means that $A(v)$ is contained in a fixed compact set.

(R) For all $(p,(v_j, y_j)) \quad S \quad {}_j Gr[bd\,Y_j]$, if $p \quad y_j \quad 0$, $j \quad \{1,\ldots, n\}$ then $r_i(p,(y_j)) > 0$, for all $i \quad \{1,\ldots, m\}$.

(iv) Hypotheses on the parameters.

(V) V is convex and compact.

Remark 3.3. Brief discussion on the hypotheses

i.– To consider $X_i = \mathbb{R}^\ell_+$ is not restrictive for the model. A more general assumption could be consider X_i as a non-empty, convex, closed and bounded below subset of $\mathbb{R}^\ell$ as we can see in Bonnisseau and Cornet (1988b.). We assume this hypothesis because of simplicity. Hypotheses on utility function and wealth are common in the literature.

ii.– For production set (mapping) we are assuming free-disposal assumption and impossibility of free production (Debreu, 1959). Finally, (PR) and (NL) are standard in the literature.

iii.– This hypothesis is very standard in the literature. The only comment is that it is required for all the images of the production set valued map.

iv.– This is the strongest condition we required for the existence result: all of other conditions are very standard in the literature. The main criticism could come from the convexity of V and not from compactness of it. In particular, under convexity we are enforced to assume a continuum of technological alternatives for every firm which could valid just for special industries and not for the generality. To consider discreet decision variables is out of the model and it is a more complicated problem that has no chance in the presented schedule. In order to incorporate discrete decision variables in the model (more generally, existence of indivisibilities in the economy), economical theory has given a partial answer to the problem of existence of equilibrium. In fact, perfect divisibility of commodities is one of the crucial assumptions in the model and corresponds to an idealized representation of a commodity space. See Bobzin (1998) for a survey in this field.

The main result of this paper is the following theorem, whose demonstration is directly inspired in the proof of existence of equilibrium given in Bonnisseau and Cornet (1988b.)[4].

Theorem 3.1. *Under assumptions V, C, B, P, SA, R, PR and NL the economy* E_V *has an equilibrium point.*

Proof. To prove this result, we will take some ideas from Bonnisseau and Cornet (1988a.), and Bonnisseau and Cornet (1988b.). Thus, let $e = (1,\dots,1)$ $\mathbb{R}^\ell$

4 Author want to thanks specially Jean-Marc Bonnisseau for helpful comments in this part.

and e the orthogonal space to it. Given $j \quad \{1,..., n\}, v_j \quad V$ and $s \quad e$, from hypothesis **P** and Lemma 5.1 of Bonnisseau and Cornet (1988b.) we already know that there exists a unique real number $_j(v_j, s)$ such that $s \quad _j(v_j, s)e \quad bd[Y_j(v_j)]$.

From the same result we also know that the mapping

$$^{v_j}_j : e \qquad bd[Y_j(v_j)]$$

$^{v_j}_j(s) = s \quad _j(v_j, s)e$ is an homeomorphism. Moreover, from the hypotheses and, it can shown that this map is continuous with respect to $v_j \quad V$.

From hypothesis **B** we have that there exists a compact set K_1 (independent of V) such that for each $v_j \quad V$ the set of attainable production plans $\widehat{Y}_j(v_j)$ is contained in K_1.

Thus, from a similar argument used in Bonnisseau and Cornet (1988b.) we have that there exist a closed ball $\overline{B} \quad (e \quad)^n$ such that for all $(v_j) \quad V^n$, given $(y_j) \qquad _j \widehat{Y}_j(v_j)$,

$$proj_{(e \quad)^n}((y_j)) \quad int\overline{B}.$$

Let $k \quad \mathbb{N}$ and let $X_i^k := X_i \bigcap [\{ke\} \quad \mathbb{R}^\ell_+]$. In fact, from X_i definition, $X_i^k = \mathbb{R}^\ell_+ \bigcap [\{ke\} \quad \mathbb{R}^\ell_+]$. Clearly X_i^k is a compact set.

On other hand, from the definition of X_i and hypothesis **B** we have that there exists a constant $\quad > 0$ such that for every $x_i \quad \widehat{X}_i$ (the attainable consumption set for individual i), $0 \quad x_i \ll \quad e$.

Given $i \quad \{1,..., m\}$, let $_i(p, \quad) = \{x_i \quad X_i \mid p \quad x_i \quad \}$, where $p \quad \mathbb{R}^\ell$ and $\mathbb{R}$. With this, we define

$$_i(p, \quad) := \{x_i \qquad _i(p, \quad) \mid u_i(x) \quad u_i(x_i) \quad x \qquad _i(p \quad)\}$$

and the set valued map f_i such that

$$f_i(p, \quad) = \begin{cases} _i(p, \quad) & if \quad > 0, \\ \{x_i \quad X_i \mid p \quad x_i = 0\} & if\ not \end{cases}$$

which will be called the demand of individual i. From hypotheses **C** and **B**, we can readily deduce that f_i is u.s.c., compact and convex valued.

Given $\quad > 0$, we set $S := \{p \quad \mathbb{R}^\ell \mid \sum_{j=1}^{\ell} p_j = 1, p_j \quad \}$. If $\quad$ represents the projection mapping from $\mathbb{R}^\ell$ to S, let us define the following set valued map $F = \prod_{t=1}^{4} F_t$ from $\quad _i X_i \quad \overline{B} \quad S \quad S \quad V^n$ to itself, where

- $F_1(x,s,p,(p_j),(v_j)) = \quad {}_i\, f_i \ (p, r_i(\ (p),(y_j(s_j,v_j))))$
- $F_2(x,s,p,(p_j),(v_j)) = \{(\quad {}_j) \quad \bar{B} | \quad {}_j(p \quad p_j) \ (\quad {}_j \quad {}_j) \quad 0, \quad (\quad {}_j) \quad \bar{B}\}$
- $F_3(x,s,p,(p_j),(v_j)) = \{q \quad S \mid (q \quad q) \left(x \quad {}_j\, y_j(s_j,v_j) \quad w\right) \quad 0, \quad q \quad S\}$
- $F_4(x,s,p,(p_j),(v_j)) = \quad ((v_j, y_j(s_j,v_j)))$.

Then, from hypotheses **PR** we deduce that F is u.s.c., with nonempty, convex, compact values. Because of hypothesis **V** we actually have that the domain of this set valued mapping is convex and then, from Kakutani's Theorem, there exists a fixed point for this map. Let us denote this fixed point by $((x_i),s \ = (s_j), p \ ,(p_j,v_j)) \quad {}_i\, X_i \quad \bar{B} \quad S \quad (S \quad V)^n$.

Thus, we have that:

(a) $x_i \quad f_i \ (p, r_i(\ (p \ ) \ (y_j)))$, for all i.

(b) ${}_j(p \quad p_j) \ s_j \quad {}_j(p \quad p_j) \ s_j$, for all j and $(s_j) \quad \bar{B}$.

(c) $p \quad {}_i\, x_i \quad {}_j\, y_j \quad w \quad p \quad {}_i\, x_i \quad {}_j\, y_j \quad w$, for all $p \quad S$.

(d) $((v_j, p_j)) \quad ((v_j, y_j))$.

Now, if we define $y_j = s_j \quad {}_j(v_j, s_j)e)$, we will show that $((v_j), p \ ,(x_i),(y_j)) \quad V^n \quad S \quad \mathbb{R}_+^{\ell m} \quad \mathbb{R}^{\ell n}$ is an equilibrium point for the economy E_V.

In fact, from the definition of F_2 and hypothesis **NL**, we deduce that for all j, $p \quad y_j \quad 0$ which, from **R**, implies that for every i, $r_i(p \ ,(y_j)) > 0$. In consequence, from the definition of f_i ,

$$\sum_{i=1}^{m} p \quad x_i \quad p \quad \sum_{j=1}^{n} y_j \quad + w$$

By other hand, from the definition of F_3, we have that ${}_i\, x_i \quad {}_j\, y_j \quad w \quad \{0\} \quad R_{++}^{\ell}$. This implies that $((x_i),(y_j))$ is an attainable allocation.

Now, from the fact that $x_i \quad X_i$, the demand set valued map definition and the properties of the utility function, we have that $p \quad x_i \ = r_i(p \ ,(y_j))$ which implies $p \ (\sum_{i=1}^{m} x_i \quad \sum_{j=1}^{n} y_j \quad w) = 0$.

Finally, from the fact that $(y_j) \quad A(v \ )$ one has $(s_j) \quad int\bar{B}$ and then, from the definition of F_2 we conclude that $p \ = p_j \quad S$, for all j. Hence, using the definition of the set valued map F_4, one may conclude that $((v_j, p_j)) \quad ((v_j, y_j))$.

Since ${}_i\, x_i \quad {}_j\, y_j \quad w \quad \{0\} \quad R_{++}^{\ell}, p \ (\sum_{i=1}^{m} x_i \quad \sum_{j=1}^{n} y_j \quad w) = 0$ and $p \quad S$, we may conclude that

$$\sum_i x_i = \sum_j y_j + w.$$

Usual arguments in this field show that x_i maximize utility on the whole budget set.

With this last result we end the demonstration of the existence of an equilibrium point for this economy.

References

Arrow, K.J. and G. Debreu (1964). “Existence of an equilibrium for a competitive economy”. *Econometrica* 22.

Bobzin, H. (1998). *Indivisibilities: Microecomic Theory with Respect to Indivisible Goods and Factors*. Physica Verlag, Heidelberg.

Bonnisseau, J.M. and B. Cornet (1988a.). “Valuation of equilibrium and Pareto optimum in nonconvex economies”. *Journal of Mathematical Economics* 17.

Bonnisseau, J.M. and B. Cornet (1988b.). “Existence of equilibria when firms follow bounded losses pricing rules”. *Journal of Mathematical Economics* 17.

Brown, D. (1991). “Equilibrium analysis with nonconvex technologies”. *Handbook of mathematical economics*, Vol. IV. Edited by W. Hildenbrand and H. Sonnenschein. North Holland.

Clarke, F. (1983). *Optimization and nonsmooth analysis.* Wiley, New York.

Debreu, G. (1959). *Theory of value, an axiomatic analysis of economic equilibrium.* John Wiley, New York.

Jofré, A., and J. Rivera (2002). “A nonconvex separation property and applications”. To appear in *Mathematical Programing*, 2002.

Quinzii, M. (1992). *Increasing returns and efficiency.* Oxford Univ. Press.

Rockafellar, R.T. and R. Wets (1998). *Variational Analysis*. Springer.