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Incentives in Three-Sided Markets

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INCENTIVES IN THREE-SIDED MARKETS

JORGE ARENAS AND JUAN PABLO TORRES-MARTÍNEZ

Abstract. In a class of three-sided matching problems in which the core is non-empty, we show that no stable mechanism is strategy-proof for those who internalize the trilateral structure of the market in their preferences. This impossibility is related to the incompatibility between stability, one-sided strategy-proofness, and one-sided non-bossiness in two-sided markets. Furthermore, unlike what happens in marriage markets, strong restrictions on preferences are needed to ensure that stability and one-sided strategy-proofness are compatible for each market side.

Keywords: Three-sided Matching Markets - Stability - Strategy-proofness
JEL Classification: D47, C78.

1. Introduction

Consider a platform for assigning university students to exchange programs. Although this type of platform opens up room for improvements in well-being, its rules must be designed in such a way that students, exchange programs, and universities have incentives to participate and reveal truthful information. Suppose that students have preferences for exchange programs, who in turn have preferences for universities (it is only through their recommendations that exchange programs are able to obtain signals of students' quality). Each university has preferences defined on the set of pairs of exchange programs and students, although it prioritizes the former to decide whether or not to recommend someone (universities seek to strengthen their network with other institutions). In this context, it is natural to wonder if there is a way to assign to each student a place in an exchange program and a university that recommends him in such a way that no group left the platform. Moreover, it is important to analyze the incentives that participants have to reveal

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information about their preferences. In this paper, these issues are analyzed from the perspective of matching theory.[1]

We consider three-sided matching problems with mixed preferences (cf., Zhang and Zhong (2021), Arenas and Torres-Martínez (2022)). It is assumed that all sides of the market—denoted by U, V, and W—have the same number of agents. Agents in U have strict preferences defined on V, agents in V have strict preferences defined on W, and agents in W have strict lexicographic preferences defined on $V \times U$. Hence, only the latter consider the trilateral structure of the market in their preferences.

Focusing our attention on *matchings*—distributions of the population in triplets conformed by agents of different sides of the market, such that every agent belongs to one and only one triplet—we characterize the existence of stable outcomes. That is, matchings in which no group of three agents may form a new triplet to improve the well-being of those who change their relevant partners.[2] Moreover, we analyze agents' incentives to reveal information about their preferences to a central planner that wants to implement a stable matching.

Although Gale and Shapley (1962) show that a stable matching always exists in a two-sided market with strict preferences, the inclusion of a third side in the market may compromise the solvability of the problem (cf., Alkan (1989)). Moreover, a stable matching may not exist even in scenarios in which agents' preferences ignore the trilateral structure of the market at the moment they evaluate their potential partners (see Lam and Plaxton (2019, 2021), Lerner (2022)). Despite the potential difficulties that the trilateral structure of the market could generate, the characteristics of agents' preferences allow us to algorithmically prove that any three-sided matching problem with mixed preferences has a non-empty set of stable outcomes, which in turn coincides with the set of matchings that are stable to coalitions' deviations, namely the core (see Theorem 1).[3] In particular, any stable matching is Pareto efficient.

Notice that, given a three-sided problem with mixed preferences, we can determine a marriage market between agents in V and W in which the latter have preferences defined on V induced by their lexicographic preferences defined on $V \times U$. Moreover, for each matching M we can determine a marriage market between agents in U and V in which the preferences of each agent in V are induced by the preferences of her W-partner in the matching M. We show that a matching M is

[1]This type of platform can be included within the class of problems studied by Block, Cantala, and Gibaja (2020), who focus on the existence of stable mechanisms that are strategy-proof for *students*. Assuming that universities and exchange programs are strategic agents, our contribution places special emphasis on understanding the compatibility between stability and truthful revelation of information by *institutions*.

[2]To capture within our model scenarios in which some agents participate in several triplets at the same time (as in the example of the platform among students, exchange programs, and universities), it is sufficient that agents' preferences for groups of individuals are *responsive* (see Roth and Sotomayor (1989)).

[3]There is a vast literature studying restrictions on preference domains in order to ensure the existence of stable matchings in three-sided problems (see, for instance, Danilov (2003); Boros, Gurvich, Jaslar, and Krasner (2004); Eriksson, Sjöstrand, and Strimling (2006); Lahiri (2009); Biró and McDermid (2010); Huang (2010); Hofbauer (2016); Zhang, Li, Fan, Shen, Shen, and Yu (2019); Zhong and Bai (2019); Lam and Plaxton (2019, 2021); Bloch, Cantala, and Gibaja (2020, 2022); Pashkovich and Poirrier (2020); Raghavan (2021); Lerner (2022)).

stable in the three-sided problem if and only if the projections of its triplets on $U \times V$ and $V \times W$ determine stable matchings in these associated marriage markets (see Proposition 1).

This result allows us to adapt classical properties of marriage markets to characterize the incentives to reveal information in three-sided problems. For instance, appealing to the results of Dubins and Freedman (1981, Theorem 9) and Roth (1982, Theorem 5), we algorithmically prove that for any three-sided matching problem with mixed preferences there is a stable mechanism that is *strategy-proof* for agents in $U \cup V$ (see Theorem 2). In other words, there is a protocol determining a stable matching from reported preferences such that—independently of the actions of other agents—no one in $U \cup V$ has incentives to misreport information.

However, no stable mechanism is strategy-proof for agents in W (see Theorem 3). That is, the trilateral structure of the market has a deep effect on the incentives of those agents that consider it in their preferences. Notice that, an agent in W may have two reasons to misreport preferences when a stable mechanism is implemented: (i) by lying she can change her partner in V to a preferred one; or (ii) by lying she keeps her partner in V but she can change the other pairs of $V \times W$ formed, which improves her situation because endogenously induces a redistribution of agents in U in order to maintain stability. Since agents in W have lexicographic preferences defined on $V \times U$, the first incentive to lie can be avoided when the triplets are formed in such a way that the pairs between agents in V and W are determined by the application of the *Gale-Shapley W-optimal stable mechanism*.[4] Indeed, in the associated marriage market between agents in V and W, this mechanism is strategy-proof for agents in W (see Dubins and Freedman (1981) or Roth (1982)). Therefore, what our Theorem 3 shows is that the second incentive to lie is unavoidable. Essentially, what is happening is that the structure of preferences guarantees that strategy-proofness for agents in W is related to one-sided strategy-proofness and one-sided non-bossiness in two-sided markets (see Proposition 2).[5] And in marriage markets, no stable mechanism is strategy-proof and non-bossy for the same side of the market (see Remark 1).

In the context of marriage markets, Ergin (2002) imposes restrictions on preference domains ensuring the existence of a stable mechanism that is one-sided strategy-proof and one-sided non-bossy. These restrictions may allow us to guarantee that a stable mechanism that is strategy-proof for agents in W exists in our context. In this direction, assuming that agents in U and W have unrestricted preferences, we prove that a mechanism based on the sequential application of Gale-Shapley deferred acceptance algorithm is stable and strategy-proof for agents in W if and only if the preference profiles of agents in V are acyclic in the sense of Ergin (2002) (see Theorem 4). Moreover, this mechanism is also Pareto efficient for agents in W if and only if the preference profiles of agents in $U \cup V$ are Ergin-acyclic (see Theorem 5). This latter finding follows from the relationship between the Pareto efficient matchings of a three-sided problem with mixed preferences

[4]This mechanism applies the *deferred acceptance algorithm* to the induced marriage market between agents in V and W assuming that agents in W make the proposals (see Gale and Shapley (1962)).

[5]Given a *marriage market* between agents of X and Y, a mechanism is *non-bossy* for agents in $A \subseteq X \cup Y$ when no agent in this set has incentives to misreport preferences in order to modify the situation of other agents without changing her own partner. A mechanism is *bossy* when it does not satisfies non-bossiness.

and the Pareto efficient outcomes of the induced marriage markets (see Proposition 3), along with the result of Ergin (2002, Theorem 1) for two-sided markets (cf., Narita (2021)).

It is well-known that focusing on incentives to reveal information in two-sided markets, there is a tension between stability and one-sided Pareto efficiency. Indeed, in a marriage market between agents in V and W, the Gale-Shapley W-optimal stable mechanism is the only mechanism that is stable and strategy-proof for agents in W (see Alcalde and Barberà (1994, Theorem 3) and Remark 1). However, it is not Pareto efficient for these agents. On the other side, assuming that agents in W choose partners, the *serial dictatorship algorithm* determines a mechanism that is both Pareto efficient and strategy-proof for agents in W (cf., Sönmez and Ünver (2011)).[6]

In three-sided markets with mixed preferences, one-sided Pareto efficiency dominates stability when it comes to finding a mechanism that is strategy-proof for agents in W. Indeed, as was pointed out above, no stable mechanism is strategy-proof for agents in W (see Theorem 3). However, using the serial dictatorship algorithm it is possible to find a mechanism that is both strategy-proof and Pareto efficient for agents in W. Furthermore, through a sequential application of serial dictatorship, we can also determine a mechanism that is both strategy-proof and Pareto efficient for agents in $U \cup V$ (see Theorem 6).

The rest of the paper is organized as follows. In Section 2 we describe the characteristics of a three-sided matching problem with mixed preferences, the stability concepts, and the properties of centralized mechanisms. The set of stable matchings is characterized in Section 3. The incentives of agents to reveal information when a stable mechanism is implemented are analyzed in Section 4. In Section 5 we determine restrictions over preference profiles to ensure the existence of a mechanism that is stable and strategy-proof for those that consider the trilateral structure of the market in their preferences. The existence of mechanisms that are both Pareto efficient and strategy-proof for a side of the market is analyzed in Section 6. Some proofs are collected in an Appendix.

2. Three-sided matching problems with mixed preferences

A *three-sided matching problem with mixed preferences*, represented by $[U, V, W, (\succ_h)_{h\in H}]$, is characterized by a set of agents $H = U \cup V \cup W$ and a preference profile $(\succ_h)_{h\in H}$ such that:

- The sets U, V, and W are disjoint and satisfy $|U| = |V| = |W|$.
- For each $u \in U$, $\succ_u$ is a linear order defined on V.[7]
- For each $v \in V$, $\succ_v$ is a linear order defined on W.
- For each $w \in W$, $\succ_w$ is a VU-lexicographic linear order defined on $V \times U$. That is, there is a linear order $\succ_{V,w}$ defined on V and a linear order $\succ_{U,w}$ defined on U such that

$$(v, u) \succ_w (v', u') \qquad \Longleftrightarrow \qquad [\, v \succ_{V,w} v' \,] \quad \text{or} \quad [\, v = v' \text{ and } u \succ_{U,w} u' \,].$$

  We refer to $(\succ_{V,w}, \succ_{U,w})$ as the linear orders representing $\succ_w$.

[6] After determining an order for agents in W, this mechanism sequentially assigns to each of them the best alternative among those available.

[7] A *linear order* is a complete, transitive, and strict preference relation.

Let $\mathcal{R}$ be the set of preference profiles $(\succ_h)_{h\in H}$ that satisfy the properties above.

A *matching* is a set $M \subseteq U \times V \times W$ such that any $h \in H$ belongs to one and only one triplet in M. Let $\mathcal{M}$ be the set of matchings. If a triplet (u, v, w) belongs to $M \in \mathcal{M}$, then the *relevant partners* of each member are denoted by $M(u) = v$, $M(v) = w$, and $M(w) = (v, u)$.

Definition 1. A matching M is *blocked by a triplet* $(u, v, w) \in U \times V \times W$ when the following three conditions are satisfied:

$$v \succ_u M(u) \text{ or } v = M(u); \qquad w \succ_v M(v) \text{ or } w = M(v); \qquad (v, u) \succ_w M(w).$$

A matching is *stable* when it cannot be blocked by any triplet.

Hence, in a stable matching no group of three agents of different sides of the market may deviate, forming a new triplet to improve the well-being of members who change their relevant partners.

We refer to any non-empty set $A \subseteq H$ such that $|A \cap U| = |A \cap V| = |A \cap W|$ as a *coalition*. Moreover, a set $N \subseteq (A \cap U) \times (A \cap V) \times (A \cap W)$ is a *matching among the members of A* when any agent $a \in A$ belongs to one and only one triplet in N.

Definition 2. A matching M is *blocked by a coalition* $A \subseteq H$ when there exists a matching N among the members of A such that:

- For any $a \in A \cap (U \cup V)$, $N(a) \succ_a M(a)$ or $N(a) = M(a)$.
- For any $a \in A \cap W$, we have that $N(a) \succ_a M(a)$.

The *core* is the set of matchings that cannot be blocked by any coalition.

Notice that, when a matching belongs to the core, no coalition of agents can form new triplets among its members to improve the well-being of those who change their relevant partners.

Definition 3. Given a set of agents $A \subseteq H$, a matching $M \in \mathcal{M}$ is *Pareto efficient for A* when there is no $N \in \mathcal{M}$ satisfying the following properties:

- For any $a \in A$, $N(a) \succ_a M(a)$ or $N(a) = M(a)$.
- There exists $a \in A$ such that $N(a) \succ_a M(a)$.

By definition, any matching in the core is Pareto efficient for H. Notice that, if $M \in \mathcal{M}$ is Pareto efficient for A, then it is not necessarily Pareto efficient for proper subsets of A.

We refer to any function $\Phi : \mathcal{R} \to \mathcal{M}$ as a *mechanism*.

Definition 4. Given a mechanism $\Phi : \mathcal{R} \to \mathcal{M}$, it is said that:

- Φ is *stable* when the matching $\Phi[(\succ_h)_{h\in H}]$ is stable in $[U, V, W, (\succ_h)_{h\in H}]$ for any preference profile $(\succ_h)_{h\in H} \in \mathcal{R}$.
- Φ is *core-selecting* when the matching $\Phi[(\succ_h)_{h\in H}]$ belongs to the core of $[U, V, W, (\succ_h)_{h\in H}]$ for any preference profile $(\succ_h)_{h\in H} \in \mathcal{R}$.

- Given $A \subseteq H$, Φ is *Pareto efficient for A* when for any preference profile $(\succ_h)_{h\in H} \in \mathcal{R}$ the matching $\Phi[(\succ_h)_{h\in H}]$ is Pareto efficient for A in $[U, V, W, (\succ_h)_{h\in H}]$.
- Given $A \subseteq H$, Φ is *strategy-proof for A* when there is no agent $a \in A$ such that, for some preference profiles $(\succ_h)_{h\in H}, (\succ'_h)_{h\in H} \in \mathcal{R}$,

$$\Phi[(\succ_h)_{h\neq a}, \succ'_a](a) \succ_a \Phi[(\succ_h)_{h\in H}](a).$$

Notice that, if $\mathcal{G} = [\mathcal{R}, \Phi]$ is the non-cooperative game in which agents report preferences $(\succ_h)_{h\in H}$ and the matching $\Phi[(\succ_h)_{h\in H}]$ is implemented, then Φ is strategy-proof for $A \subseteq H$ if and only if in the game $\mathcal{G}$ it is a dominant strategy for agents in A to report their true preferences.

3. The set of stable matchings

In this section, we characterize the set of stable outcomes of a three-sided matching problem with mixed preferences. Our first result shows that any problem $[U, V, W, (\succ_h)_{h\in H}]$ has a stable matching and guarantees the equivalence between the core and the set of stable matchings (cf., Roth and Sotomayor (1989, Proposition 1)). This last result implies that a mechanism is stable if and only if it is core-selecting.

To find a stable matching for $[U, V, W, (\succ_h)_{h\in H}]$, we will consider a stable outcome $f : V \to W$ of the marriage market $[V, W, (\succ_v)_{v\in V}, (\succ_{V,w})_{w\in W}]$ and a stable outcome of the induced marriage market $[U, V, (\succ_u)_{u\in U}, (\succ_{U,f(v)})_{v\in V}]$, where $(\succ_{V,w}, \succ_{U,w})$ are the linear orders representing $\succ_w$. The existence of stable matchings in these two-sided problems is guaranteed by Gale and Shapley (1962).

Theorem 1. *Any three-sided matching problem with mixed preferences has a stable matching. Moreover, the set of stable matchings coincides with the core.*

Proof. Let $[U, V, W, (\succ_h)_{h\in H}]$ be a three-sided matching problem with mixed preferences. For each $w \in W$, let $(\succ_{V,w}, \succ_{U,w})$ be the linear orders representing $\succ_w$. It follows from Gale and Shapley (1962, Theorem 1) that the marriage market $[V, W, (\succ_v)_{v\in V}, (\succ_{V,w})_{w\in W}]$ has a stable matching. That is, there exists a bijective function $f : V \to W$ such that there is no $(v, w) \in V \times W$ satisfying $w \succ_v f(v)$ and $v \succ_{V,w} f^{-1}(w)$.

Let $Z = \{(v, w) \in V \times W : w = f(v)\}$. Given $z = (v, w) \in Z$, let $\succ^*_z$ be the linear order defined on U such that $u \succ^*_z u'$ if and only if $u \succ_{U,w} u'$. Moreover, given $u \in U$, let $\succ^*_u$ be the linear order defined on Z such that $z \succ^*_u z'$ if and only if $v \succ_u v'$, where $z = (v, w)$ and $z' = (v', w')$. Since Gale and Shapley (1962) ensures that $[U, Z, (\succ^*_h)_{h\in U\cup Z}]$ has a stable matching, there exists a bijective function $g : U \to Z$ such that there is no $(u, z) \in U \times Z$ satisfying $z \succ^*_u g(u)$ and $u \succ^*_z g^{-1}(z)$.

We claim that $M = \{(u, v, w) \in U \times V \times W : g(u) = (v, w)\}$ is stable in $[U, V, W, (\succ_h)_{h\in H}]$. By contradiction, suppose that (u^*, v^*, w^*) blocks M. Then, it follows from Definition 1 that the following conditions hold:

(*a*) $v^* \succ_{u^*} M(u^*)$ or $v^* = M(u^*)$.
(*b*) $w^* \succ_{v^*} f(v^*)$ or $w^* = f(v^*)$.
(*c*) $(v^*, u^*) \succ_{w^*} (f^{-1}(w^*), g^{-1}(f^{-1}(w^*), w^*))$.

Since $\succ_{w^*}$ is a VU-lexicographic linear order, the condition (c) is equivalent to requiring that either $v^* \succ_{V,w^*} f^{-1}(w^*)$ or $[v^* = f^{-1}(w^*)$ and $u^* \succ_{U,w^*} g^{-1}(f^{-1}(w^*), w^*)]$. Hence, as f determines a stable matching of $[V, W, (\succ_v)_{v\in V}, (\succ_{V,w})_{w\in W}]$, it follows from conditions (b)-(c) that

(b') $w^* = f(v^*)$.

(c') $u^* \succ_{U,w^*} g^{-1}(v^*, w^*)$.

If we denote $z^* = (v^*, w^*)$, then (c') is equivalent to $u^* \succ^*_{z^*} g^{-1}(z^*)$. Since g determines a stable matching of $[U, Z, (\succ^*_h)_{h\in U\cup Z}]$, and $v^* \succ_{u^*} M(u^*)$ is equivalent to $z^* \succ^*_{u^*} g(u^*)$, it follows from conditions (a) and (b') that $v^* = M(u^*)$ and $f(v^*) = w^*$. Hence $(u^*, v^*, w^*) \in M$, which is a contradiction. Therefore, the matching M is stable in $[U, V, W, (\succ_h)_{h\in H}]$.

We will prove that the core coincides with the set of stable matchings.

Notice that, if a matching M is not stable, then it is blocked by some triplet (u, v, w). Thus, M can be blocked by the coalition $\{u, v, w\}$ through the matching $N = \{(u, v, w)\}$. Hence, we conclude that the core is a subset of the collection of stable matching.

On the other hand, by contradiction, assume that M is a stable matching that does not belong to the core. Then, for some coalition A there is a matching N among its members such that $N(w) \succ_w M(w)$ for all $w \in A \cap W$. Moreover, if $(u, v, w) \in N$, then one of the next properties hold:

$$v \succ_u M(u) \text{ and } w \succ_v M(v), \qquad v = M(u) \text{ and } w \succ_v M(v), \qquad v \succ_u M(u) \text{ and } w = M(v).$$

Therefore, the matching M is blocked by the triplet (u, v, w), which contradicts its stability. □

To ensure that a problem $[U, V, W, (\succ_h)_{h\in H}]$ has a stable matching it was crucial to restrict the preferences of agents in W to the domain of VU-lexicographic preferences. Indeed, without this assumption, and by appealing to the result of Lam and Plaxton (2019) regarding the non-existence of stable outcomes in three-sided matching problems with *cyclic preferences*, Arenas and Torres-Martínez (2022) show that $[U, V, W, (\succ_h)_{h\in H}]$ may have an empty set of stable outcomes.[8]

We wish to gain a deeper understanding of the relationship between a stable matching M of the three-sided problem with mixed preferences $[U, V, W, (\succ_h)_{h\in H}]$ and the stable matchings of two-sided problems $[V, W, (\succ_v)_{v\in V}, (\succ_{V,w})_{w\in W}]$ and $[U, V, (\succ_u)_{u\in U}, (\succ_{U,M(v)})_{v\in V}]$.

Let $\theta : \mathcal{M} \twoheadrightarrow U \times V$ and $\psi : \mathcal{M} \twoheadrightarrow V \times W$ be the correspondences characterized by

$$\theta(M) = \{(u, v) \in U \times V : M(u) = v\}, \qquad \psi(M) = \{(v, w) \in V \times W : M(v) = w\}.$$

That is, $\theta(M)$ and $\psi(M)$ are the projections of M on the sets $U \times V$ and $V \times W$, respectively.

The following result characterizes the structure of the set of stable matchings of a three-sided matching problem with mixed preferences.

[8]Lam and Plaxton (2019, 2021) disproved the conjecture of Eriksson, Sjöstrand, and Strimling (2006) about the existence of stable matchings in three-sided problems with *cyclic preferences* (cf., Lerner (2022)).

Proposition 1. *Given a three-sided matching problem* $[U, V, W, (\succ_h)_{h\in H}]$*, a matching* M *is stable in* $[U, V, W, (\succ_h)_{h\in H}]$ *if and only if the following properties hold:*

(i) *The matching* $\theta(M)$ *is stable in the marriage market* $[U, V, (\succ_u)_{u\in U}, (\succ_{U,M(v)})_{v\in V}]$.

(ii) *The matching* $\psi(M)$ *is stable in the marriage market* $[V, W, (\succ_v)_{v\in V}, (\succ_{V,w})_{w\in W}]$.

The proof is given in the Appendix.

It follows from Gale and Shapley (1962) and Proposition 1 that for any three-sided problem with mixed preferences $[U, V, W, (\succ_h)_{h\in H}]$ we can algorithmically construct a stable outcome.

Indeed, let $\mathrm{DA}_{\mathrm{W},3} : \mathcal{R} \to \mathcal{M}$ be the mechanism that associates with each preference profile $(\succ_h)_{h\in H} \in \mathcal{R}$ the matching obtained through the following procedure, which sequentially applies the Gale-Shapley *deferred acceptance algorithm*:

Step 1. Given $w \in W$, let $(\succ_{V,w}, \succ_{U,w})$ be the linear orders representing $\succ_w$.
Assuming that agents in W propose to agents in V, apply the *deferred acceptance algorithm* to the marriage market $[V, W, (\succ_v)_{v\in V}, (\succ_{V,w})_{w\in W}]$. Let $Z \subseteq V \times W$ be the set of pairs formed.

Step 2. For each $z = (v, w) \in Z$, let $\succ_z^*$ be the linear order defined on U such that $u \succ_z^* u'$ whenever $u \succ_{U,w} u'$. Moreover, for each $u \in U$, let $\succ_u^*$ be the linear order defined on Z such that $z \succ_u^* z'$ as long as $v \succ_u v'$, where $z = (v, w)$ and $z' = (v', w')$.

Step 3. Assuming that agents in Z propose to agents in U, apply the *deferred acceptance algorithm* to the marriage market $[U, Z, (\succ_h^*)_{h\in U\cup Z}]$.
Define $\mathrm{DA}_{\mathrm{W},3}[(\succ_h)_{h\in H}]$ as the set of triplets obtained.

It follows from Proposition 1 that $\mathrm{DA}_{\mathrm{W},3}$ is a stable mechanism.[9]

In two-sided matching problems, the Gale-Shapley deferred acceptance algorithm determines mechanisms that are strategy-proof for those that make proposals (cf., Dubins and Freedman (1981) or Roth (1982)). Hence, although it might be expected that $\mathrm{DA}_{\mathrm{W},3}$ is strategy-proof for agents in W, in our framework no stable mechanism will be strategy-proof for those that consider the trilateral structure of the market in their preferences (see Theorem 3 in the next section).

4. On stability and strategy-proofness

In this section, we analyze agents' incentives to reveal information about preferences. We start by showing that there is a stable mechanism that is strategy-proof for those who only consider one side of the market in their preferences.[10]

Theorem 2. *There exists a mechanism* $\Phi : \mathcal{R} \to \mathcal{M}$ *that is stable and strategy-proof for* $U \cup V$.

[9]There are other three-sided matching problems where algorithms based in the sequential application of the Gale-Shapley deferred acceptance algorithm induce stable matchings (see, for instance, Danilov (2003), Zhong and Bai (2019), Block, Cantala, and Gibaja (2020), Zhang and Zhong (2021)).

[10]This property does not hold in two-sided matching markets (cf., Roth (1982, Theorem 3)).

Proof. Let $\mathrm{DA}_{\mathrm{VU},3} : \mathcal{R} \to \mathcal{M}$ be the mechanisms that associates with each preference profile $(\succ_h)_{h\in H}$ the matching that is obtained by the following procedure:

Step 1. Given $w \in W$, let $(\succ_{V,w}, \succ_{U,w})$ be the linear orders representing $\succ_w$.
Assuming that agents in V propose to agents in W, apply the *deferred acceptance algorithm* to the marriage market $[V, W, (\succ_v)_{v\in V}, (\succ_{V,w})_{w\in W}]$. Let $Z \subseteq V \times W$ be the set of pairs formed.

Step 2. For each $z = (v, w) \in Z$, let $\succ_z^*$ be the linear order defined on U such that $u \succ_z^* u'$ whenever $u \succ_{U,w} u'$. Also, for each $u \in U$, let $\succ_u^*$ be the linear order defined on Z such that $z \succ_u^* z'$ as long as $v \succ_u v'$, where $z = (v, w)$ and $z' = (v', w')$.

Step 3. Assuming that agents in U propose to agents in Z, apply the *deferred acceptance algorithm* to the marriage market $[U, Z, (\succ_h^*)_{h\in U\cup Z}]$.
Define $\mathrm{DA}_{\mathrm{VU},3}[\succ_h)_{h\in H]}$ as the set of triplets obtained.

Given a preference profile $(\succ_h)_{h\in H} \in \mathcal{R}$, it follows from Gale and Shapley (1962, Theorem 1) and our Proposition 1 that the matching $\mathrm{DA}_{\mathrm{VU},3}[(\succ_h)_{h\in H}]$ is stable in $[U, V, W, (\succ_h)_{h\in H}]$.

Since in a two-sided market with strict preferences the deferred acceptance mechanism is strategy-proof for those that make proposals (see Dubins and Freedman (1981, Theorem 9) or Roth (1982, Theorem 5)), we conclude that $\mathrm{DA}_{\mathrm{VU},3}$ is strategy-proof for agents in $U \cup V$. □

In the context of marriage markets with strict preferences, it is well-known that there is a unique stable matching if and only if the result of the deferred acceptance algorithm does not depend on the side of the market that makes proposals. As a consequence, it follows from Proposition 1 and the proof of Theorem 2, that a three-sided problem with mixed preferences $[U, V, W, (\succ_h)_{h\in H}]$ has a unique stable matching if and only if $\mathrm{DA}_{W,3}[(\succ_h)_{h\in H}] = \mathrm{DA}_{\mathrm{VU},3}[(\succ_h)_{h\in H}]$.

Unlike what happens in two-sided matching models (Dubins and Freedman (1981), Roth (1982)), there is a side of the market such that no stable mechanism is strategy-proof for its members.

Theorem 3. *There is no stable mechanism $\Phi : \mathcal{R} \to \mathcal{M}$ that is strategy-proof for W.*

Proof. Consider a three-sided matching problem with mixed preferences $[U, V, W, (\succ_h)_{h\in H}]$ where the sets of agents are given by $U = \{u_1, u_2, u_3, u_4\}$, $V = \{v_1, v_2, v_3, v_4\}$, and $W = \{w_1, w_2, w_3, w_4\}$. Suppose that linear orders $(\succ_h)_{h\in U\cup V}$ satisfy the following conditions:[11]

$\succ_{u_1}$	$\succ_{u_2}$	$\succ_{u_3}$	$\succ_{u_4}$
v_1	v_2	v_2	v_4
⋮	v_3	v_3	⋮
⋮	⋮	⋮	⋮
⋮	⋮	⋮	⋮

$\succ_{v_1}$	$\succ_{v_2}$	$\succ_{v_3}$	$\succ_{v_4}$
w_2	w_1	w_3	w_2
w_3	w_2	⋮	w_4
w_1	⋮	⋮	⋮
w_4	⋮	⋮	⋮

[11]In the description of preferences, the vertical dots stand for arbitrary ordering of agents.

Moreover, for each $w \in W$, the linear orders $(\succ_{V,w}, \succ_{U,w})$ representing $\succ_w$ are such that

$\succ_{V,w_1}$	$\succ_{V,w_2}$	$\succ_{V,w_3}$	$\succ_{V,w_4}$
v_1	v_2	v_3	v_4
$\vdots$	v_1	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

$\succ_{U,w_1}$	$\succ_{U,w_2}$	$\succ_{U,w_3}$	$\succ_{U,w_4}$
u_1	u_1	u_2	u_4
u_3	u_2	u_3	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

In this context, it follows from Proposition 1 that the only stable matchings are

$$\begin{aligned} M &= \{(u_3, v_2, w_1), (u_1, v_1, w_2), (u_2, v_3, w_3), (u_4, v_4, w_4)\}, \\ M' &= \{(u_1, v_1, w_1), (u_2, v_2, w_2), (u_3, v_3, w_3), (u_4, v_4, w_4)\}. \end{aligned}$$

Therefore, if $\Phi : \mathcal{R} \to \mathcal{M}$ is a stable mechanism, we have two alternatives:

(i) $\Phi[(\succ_h)_{h\in H}] = M$. In this case, if all agents $h \neq w_2$ report their true preferences, then w_2 improves her situation by reporting $v_2 \succ^*_{V,w_2} v_4 \succ^*_{V,w_2} v_3 \succ^*_{V,w_2} v_1$ instead of $\succ_{V,w_2}$. Indeed, M' is the only stable matching in this scenario and $(v_2, u_2) \succ_{w_2} (v_1, u_1)$.

(ii) $\Phi[(\succ_h)_{h\in H}] = M'$. In this case, if all agents $h \neq w_3$ report their true preferences, then w_3 improves her situation by reporting $v_1 \succ^*_{V,w_3} v_3 \succ^*_{V,w_3} v_2 \succ^*_{V,w_3} v_4$ instead of $\succ_{V,w_3}$. Indeed, M is the only stable matching in this scenario and $(v_3, u_2) \succ_{w_3} (v_3, u_3)$.

We conclude that the stable mechanism Φ is not strategy-proof for W. □

To gain some intuition behind the proof of Theorem 3, given a preference profile $(\succ_h)_{h\in H} \in \mathcal{R}$, suppose that the induced marriage market $[V, W, (\succ_v)_{v\in V}, (\succ_{V,w})_{w\in W}]$ has only two stable outcomes, μ and μ'. Without loss of generality, assume that μ is the V-optimal stable matching and μ' is the W-optimal stable matching.[12]

In this context, Gale and Sotomayor (1985, Theorem 1) ensures that at least one agent in W has incentives to misreport preferences when μ is implemented. Since in our framework agents in W have VU-lexicographic preferences, it is natural that an analogous property holds: given a stable mechanism $\Phi : \mathcal{R} \to \mathcal{M}$, at least one agent of W should have incentives to misreport preferences when $\Phi[(\succ_h)_{h\in H}]$ is such that $\Phi[(\succ_h)_{h\in H}](v) = \mu(v)$ for all $v \in V$. This is exactly what happens in the proof of Theorem 3 (see item (i)).

Therefore, as a consequence of Proposition 1, and to have any chance that Φ is strategy-proof for W, the projection of $\Phi[(\succ_h)_{h\in H}]$ on $V \times W$ needs to be equal to μ'. However, if an agent $w \in W$ has the same partner in μ and μ', she may have incentives to misreport preferences in order to change the other couples of $V \times W$ without modifying her partner on V. Indeed, with this action she may improve her situation, by reducing the interest of some agents in U for the other pairs of $V \times W$. This is exactly what happens in the proof of Theorem 3 (see item (ii)).

Intuitively, it seems that strategy-proofness for W is related to one-sided strategy-proofness and one-sided non-bossiness in two-sided matching markets (see Proposition 2).

[12]That is, μ is weakly preferred by every agent in V to any other stable matching of $[V, W, (\succ_v)_{v\in V}, (\succ_{V,w})_{w\in W}]$, and the analogous property holds for μ' with respect to agents in W (see Gale and Shapley (1962, Theorem 2)).

In two-sided matching problems, each side of the market has an optimal stable outcome (Gale and Shapley (1962, Theorem 2)) and the set of stable matchings has a lattice structure (cf., Knuth (1976)). These properties are lost in our context. Indeed, in the matching problem described in the proof of Theorem 3 there are only two stable outcomes, and none of them match each agent in W with the best partner that she may have in a stable matching.[13]

5. Existence of stable and strategy-proof mechanisms

In this section, we formalize the intuition that strategy-proofness for W in three-sided markets with mixed preferences is related to one-sided strategy-proofness and one-sided non-bossiness in marriage markets. We will use this result to determine restrictions over the domain of preferences in order to ensure that the stable mechanism $\mathrm{DA}_{\mathrm{W},3}$ is strategy-proof for agents in W.

To achieve these objetives, we need to introduce notation for the preference domains of the marriage markets induced by a three-sided market. Let $\mathcal{S}$ be the set of profiles $(S_h)_{h\in U\cup V}$ such that, for every $(u,v)\in U\times V$, S_u is a linear order defined on V and S_v is a linear order defined on U. Moreover, let $\mathcal{Q}$ be the set of profiles $(Q_h)_{h\in V\cup W}$ such that, for every $(v,w)\in V\times W$, Q_v is a linear order defined on W and Q_w is a linear order defined on V.

Definition 5. Given mechanisms $\Theta:\mathcal{S}\to\theta(\mathcal{M})$ and $\Psi:\mathcal{Q}\to\psi(\mathcal{M})$, it is said that:

- Θ is *strategy-proof for V* when there is no agent $v\in V$, preference profile $S\in\mathcal{S}$, and linear order S'_v defined on U such that $\Theta[S_{-v},S'_v](v)\, S_v\, \Theta[S](v)$.
- Ψ is *strategy-proof for W* when there is no agent $w\in W$, preference profile $Q\in\mathcal{Q}$, and linear order Q'_w defined on V such that $\Psi[Q_{-w},Q'_w](w)\, Q_w\, \Psi[Q](w)$.
- Ψ is *non-bossy for W* when there is no agent $w\in W$, preference profile $Q\in\mathcal{Q}$, and linear order Q'_w defined on V such that $\Psi[Q_{-w},Q'_w](w)=\Psi[Q](w)$ and $\Psi[Q_{-w},Q'_w]\neq\Psi[Q]$.

Hence, $\Psi:\mathcal{Q}\to\psi(\mathcal{M})$ is non-bossy for W as long as no $w\in W$ has incentives to misreport preferences in order to modify the situation of other agents without changing her own partner.

Notice that, for every preference profile $(\succ_h)_{h\in H}\in\mathcal{R}$ and for any matching N between the members in V and W, if $(\succ_{V,w},\succ_{U,w})$ are the linear orders representing $\succ_w$, then

$$((\succ_u)_{u\in U},(\succ_{U,N(v)})_{v\in V})\in\mathcal{S}\qquad\wedge\qquad((\succ_v)_{v\in V},(\succ_{V,w})_{w\in W})\in\mathcal{Q}.$$

Therefore, given $\Theta:\mathcal{S}\to\theta(\mathcal{M})$ and $\Psi:\mathcal{Q}\to\psi(\mathcal{M})$, we can define the mechanism $\Phi_{\Theta,\Psi}:\mathcal{R}\to\mathcal{M}$ that associates with each profile $(\succ_h)_{h\in H}$ the set of triplets (u,v,w) such that

$$\Theta[(\succ_u)_{u\in U},(\succ_{U,N(v)})_{v\in V}](u)=v=\Psi[(\succ_v)_{v\in V},(\succ_{V,w})_{w\in W}](w),$$

where $N=\Psi[(\succ_v)_{v\in V},(\succ_{V,w})_{w\in W}]$. It follows from Proposition 1 that $\Phi_{\Theta,\Psi}$ is stable if and only if the mechanisms Θ and Ψ are stable. Moreover, Theorem 3 guarantees that $\Phi_{\Theta,\Psi}$ cannot be stable and strategy-proof for W in the whole preference domain $\mathcal{R}$.

[13] While agents w_1 and w_2 prefer M' to M, agent w_3 prefers M to M'.

We refer to a sub-domain $\mathcal{R}' \subseteq \mathcal{R}$ as UW-*unrestricted* when for every specification of $(\succ_h)_{h\in U\cup W}$, there exists $(\succ'_v)_{v\in V}$ such that

$$((\succ_u)_{u\in U}, (\succ'_v)_{v\in V}, (\succ_w)_{w\in W}) \in \mathcal{R}'.$$

Hence, $\mathcal{R}'$ is UW-unrestricted when only the linear orders $(\succ_v)_{v\in V}$ are constrained.

Given a sub-domain $\mathcal{R}' \subseteq \mathcal{R}$, let $\mathcal{Q}(\mathcal{R}') \subseteq \mathcal{Q}$ be the set of preference profiles $(\succ_v, \succ_{V,w})_{(v,w)\in V\times W}$ that are considered in $\mathcal{R}'$. Equivalently, $\mathcal{Q}(\mathcal{R}')$ is the collection of $(Q_h)_{h\in V\cup W} \in \mathcal{Q}$ such that

$$((\succ_u)_{u\in U}, (Q_v)_{v\in V}, (Q_w, \succ_{U,w})_{w\in W}) \in \mathcal{R}'$$

for some linear orders $(\succ_u)_{u\in U}$ and $(\succ_{U,w})_{w\in W}$.

The next result determines necessary and sufficient conditions to ensure that the mechanism $\Phi_{\Theta,\Psi}$ is strategy-proof for W in a sub-domain $\mathcal{R}' \subseteq \mathcal{R}$.

Proposition 2. *Let $\mathcal{R}' \subseteq \mathcal{R}$ be a* UW-*unrestricted sub-domain. Given two stable mechanisms $\Theta : \mathcal{S} \to \theta(\mathcal{M})$ and $\Psi : \mathcal{Q}(\mathcal{R}') \to \psi(\mathcal{M})$, the following properties are equivalent:*

(i) *The mechanism $\Phi_{\Theta,\Psi} : \mathcal{R}' \to \mathcal{M}$ is strategy-proof for W.*

(ii) *Θ is strategy-proof for V, while Ψ is both strategy-proof for W and non-bossy for W.*

The proof is given in the Appendix.

In the context of a marriage market between V and W, the *Gale-Shapley W-optimal stable mechanism* associates with every profile of preferences $(Q_h)_{h\in V\cup W} \in \mathcal{Q}$ the outcome of the deferred acceptance algorithm when agents in W make proposals.

Given a UW-unrestricted sub-domain $\mathcal{R}' \subseteq \mathcal{R}$, it follows from Ergin (2002, Theorem 1) and Pápai (2000, Lemma 1) that the Gale-Shapley W-optimal stable mechanism is both strategy-proof for W and non-bossy for W in the sub-domain $\mathcal{Q}(\mathcal{R}')$ if and only if the preferences of agents in V satisfy the following condition.

Definition 6. A preference profile $(\succ_v)_{v\in V}$ is *Ergin-acyclic* when there are no agents $v_1, v_2 \in V$ and $w_1, w_2, w_3 \in W$ such that $w_1 \succ_{v_1} w_2 \succ_{v_1} w_3$ and $w_3 \succ_{v_2} w_1$.

A sub-domain $\mathcal{R}' \subseteq \mathcal{R}$ is V-*Ergin-acyclic* when for every $(\succ_h)_{h\in H} \in \mathcal{R}'$ the preference profile $(\succ_v)_{v\in V}$ is Ergin-acyclic.

Notice that, in any of the three-sided matching problems considered in the proof of Theorem 3 the preference profile $(\succ_v)_{v\in V}$ is Ergin-cyclic, because $w_2 \succ_{v_4} w_4 \succ_{v_4} w_3$ and $w_3 \succ_{v_3} w_2$. Hence, it seems that V-Ergin-acyclicity is a necessary condition to ensure existence of a stable mechanism that is strategy-proof for W. The following result formalizes this idea.

Theorem 4. *If* $\mathcal{R}' \subseteq \mathcal{R}$ *is* UW*-unrestricted, then the following properties are equivalent:*

(i) $\text{DA}_{\text{W},3} : \mathcal{R}' \to \mathcal{M}$ *is strategy-proof for* W.
(ii) $\mathcal{R}'$ *is* V*-Ergin-acyclic.*

Proof. Let $\Theta : \mathcal{S} \to \theta(\mathcal{M})$ be the mechanism that associates with each $(S_h)_{h\in U\cup V}$ the V-optimal stable matching of the marriage market $[U, V, (S_h)_{h\in U\cup V}]$. Also, let $\Psi : \mathcal{Q}(\mathcal{R}') \to \psi(\mathcal{M})$ be the mechanism that associates with each $(Q_h)_{h\in V\cup W} \in \mathcal{Q}(\mathcal{R}')$ the W-optimal stable matching of the marriage market $[V, W, (Q_h)_{h\in V\cup W}]$. Notice that $\text{DA}_{\text{W},3} \equiv \Phi_{\Theta,\Psi}$ in the sub-domain $\mathcal{R}'$.

$[(i) \implies (ii)]$ Suppose that the mechanism $\text{DA}_{\text{W},3} : \mathcal{R}' \to \mathcal{M}$ is strategy-proof for W. Since $\mathcal{R}' \subseteq \mathcal{R}$ is UW-unrestricted, Proposition 2 ensures that $\Psi : \mathcal{Q}(\mathcal{R}') \to \psi(\mathcal{M})$ is both strategy-proof for W and non-bossy for W. Hence, it follows from Ergin (2002, Theorem 1) that, for any preference profile $(Q_h)_{h\in V\cup W} \in \mathcal{Q}(\mathcal{R}')$, the linear orders $(Q_v)_{v\in V}$ are Ergin-acyclic.[14] Therefore, the sub-domain $\mathcal{R}'$ is V-Ergin-acyclic.

$[(ii) \implies (i)]$ When $\mathcal{R}'$ is UW-unrestricted and V-Ergin-acyclic, Ergin (2002, Theorem 1) ensures that $\Psi : \mathcal{Q}(\mathcal{R}') \to \psi(\mathcal{M})$ is both strategy-proof for W and non-bossy for W (cf., Narita (2021)). Moreover, it is well-known that the mechanism $\Theta : \mathcal{S} \to \theta(\mathcal{M})$ is strategy-proof for V (see Dubins and Freedman (1981, Theorem 9) or Roth (1982, Theorem 5)). Hence, Proposition 2 guarantees that $\text{DA}_{\text{W},3}$ is strategy-proof for W in the sub-domain $\mathcal{R}'$. □

Since V-acyclicity restricts substantially the heterogeneity of the preferences of agents in V, strong restrictions on $\mathcal{R}'$ are *necessary* to ensure that $\text{DA}_{\text{W},3} : \mathcal{R}' \to \mathcal{M}$ is strategy-proof for W. In other words, the result above reinforces the relevance that the trilateral structure of the market has on agents' incentives to reveal information about preferences.

Remark 1. In a marriage market between V and W, if agents may consider some potential partners inadmissible, then no stable mechanism is strategy-proof for W and non-bossy for W. In fact, in this preference domain, the Gale-Shapley W-optimal stable mechanism is the only stable mechanism that is strategy-proof for W (see Alcalde and Barberà (1994, Theorem 3)). And it is well-known that this mechanism is bossy for W (see Roth (1982, Section 6)).

The non-existence of stable mechanisms that are strategy-proof for W and non-bossy for W still holds in the restricted domain $\mathcal{Q}$. Indeed, let $\Theta : \mathcal{S} \to \theta(\mathcal{M})$ be the Gale-Shapley V-optimal stable mechanism. Since Θ is strategy-proof for V (see Dubins and Freedman (1981) or Roth (1982)), if there is $\Psi : \mathcal{Q} \to \psi(\mathcal{M})$ stable, strategy-proof for W, and non-bossy for W, then Propositions 1 and 2 imply that $\Phi_{\Theta,\Psi} : \mathcal{R} \to \mathcal{M}$ is stable and strategy-proof for W. This contradicts the result of Theorem 3 (see Proposition A1 for a constructive proof). □

[14]Although in our framework agents in W consider all members in V admissible, the main result of Ergin (2002) can be easily adapted. Admissibility plays a role only in steps $[(ii) \implies (iv)]$ and $[(iii) \implies (iv)]$ of the proof of his Theorem 1. In these steps, given $v_1, v_2 \in V$ and $w_1, w_2, w_3 \in W$, it is considered a preference profile $(Q_w)_{w\in W}$ such that $v_2 Q_{w_1} v_1 Q_{w_1} w_1 Q_{w_1} \cdots$, $v_1 Q_{w_2} w_2 Q_{w_2} \cdots$, $v_1 Q_{w_3} v_2 Q_{w_3} w_3 Q_{w_3} \cdots$, and $w Q_w v$ for all $w \in W \setminus \{w_1, w_2, w_3\}$ and $v \in V$. However, if $\{v_3, \ldots, v_n\} \equiv V \setminus \{v_1, v_2\}$ and $\{w_4, \ldots, w_n\} \equiv W \setminus \{w_1, w_2, w_3\}$, the same implications that are obtained from $(Q_w)_{w\in W}$ can be ensured by working with the linear orders $(\tilde{Q}_w)_{w\in W}$ defined on V and characterized by $v_2 \tilde{Q}_{w_1} v_1 \tilde{Q}_{w_1} \cdots$, $v_1 \tilde{Q}_{w_2} v_3 \tilde{Q}_{w_2} \cdots$, $v_1 \tilde{Q}_{w_3} v_2 \tilde{Q}_{w_3} \cdots$, and $v_j \tilde{Q}_{w_j} \cdots$ for all $j \in \{4, \ldots, n\}$.

6. On Pareto efficiency and strategy-proofness

In this section, we characterize the Pareto efficient matchings of a three-side problem with mixed preferences in terms of the efficient allocations of the associated marriage markets. As a byproduct, we determine restrictions over preference domains to ensure the existence of a *stable* mechanism that is both strategy-proof and Pareto efficient for agents in W. Furthermore, without restricting preference domains, we show that for any side of the market there is a mechanism that is both Pareto efficient and strategy-proof for its members.

Definition 7. Given a matching $M \in \mathcal{M}$, we have that:

- The matching $\theta(M)$ is *Pareto efficient for* V when there is no $N \in \mathcal{M}$ such that:
  - For any $v \in V$, either $\theta(N)(v) \succ_{U,M(v)} \theta(M)(v)$ or $\theta(N)(v) = \theta(M)(v)$.
  - There exists $v \in V$ such that $\theta(N)(v) \succ_{U,M(v)} \theta(M)(v)$.
- The matching $\psi(M)$ is *Pareto efficient for* W when there is no $N \in \mathcal{M}$ such that:
  - For any $w \in W$, either $\psi(N)(w) \succ_{V,w} \psi(M)(w)$ or $\psi(N)(w) = \psi(M)(w)$.
  - There exists $w \in W$ such that $\psi(N)(w) \succ_{V,w} \psi(M)(w)$.

We will show that a matching M is Pareto efficient for W in $[U, V, W, (\succ_h)_{h\in H}]$ if and only if its projections on $U \times V$ and $V \times W$ satisfy efficiency properties with respect to V and W in marriage markets $[U, V, (\succ_u)_{u\in U}, (\succ_{U,M(v)})_{v\in V}]$ and $[V, W, (\succ_v)_{v\in V}, (\succ_{V,w})_{w\in W}]$.

Proposition 3. *A matching M is Pareto efficient for W in the three-sided matching problem $[U, V, W, (\succ_h)_{h\in H}]$ if and only if the following properties hold:*

(i) *The matching $\theta(M)$ is Pareto efficient for V in $[U, V, (\succ_u)_{u\in U}, (\succ_{U,M(v)})_{v\in V}]$.*
(ii) *The matching $\psi(M)$ is Pareto efficient for W in $[V, W, (\succ_v)_{v\in V}, (\succ_{V,w})_{w\in W}]$.*

The proof is given in the Appendix.

Definition 8. A preference profile $(\succ_u)_{u\in U}$ is *Ergin-acyclic* when there are no agents $u_1, u_2 \in U$ and $v_1, v_2, v_3 \in V$ such that $v_1 \succ_{u_1} v_2 \succ_{u_1} v_3$ and $v_3 \succ_{u_2} v_1$.

A sub-domain $\mathcal{R}' \subseteq \mathcal{R}$ is UW-*acyclic-unrestricted* when for every $(\succ_h)_{h\in U\cup W}$ such that $(\succ_u)_{u\in U}$ is Ergin-acyclic there are linear orders $(\succ'_v)_{v\in V}$ such that $((\succ_u)_{u\in U}, (\succ'_v)_{v\in V}, (\succ_w)_{w\in W}) \in \mathcal{R}'$.

Hence, $\mathcal{R}' \subseteq \mathcal{R}$ is UW-acyclic-unrestricted when for every $(\succ_h)_{h\in H} \in \mathcal{R}'$ the linear orders $(\succ_u)_{u\in U}$ are *only* constrained to be Ergin-acyclical and the linear orders $(\succ_w)_{w\in W}$ are unconstrained.

The following result determines sufficient conditions over the domain of preferences to ensure the existence of a stable mechanism that is Pareto efficient for W and strategy-proof for W. Essentially, the linear orders representing the preferences of agents in $U \cup V$ are required to be Ergin acyclic.

Theorem 5. *If $\mathcal{R}' \subseteq \mathcal{R}$ is* UW*-acyclic-unrestricted, then the following properties are equivalent:*

(i) $\text{DA}_{\text{W},3} : \mathcal{R}' \to \mathcal{M}$ *is strategy-proof for W.*
(ii) $\mathcal{R}'$ *is* V*-Ergin-acyclic.*
(iii) $\text{DA}_{\text{W},3} : \mathcal{R}' \to \mathcal{M}$ *if Pareto efficient for W.*

Proof. Given a sub-domain $\mathcal{R}' \subseteq \mathcal{R}$ and a bijective function $f : V \to W$, let $\mathcal{S}(\mathcal{R}') \subseteq \mathcal{S}$ be the set of profiles $(\succ_u, \succ_{U,f(v)})_{(u,v)\in U\times V}$ such that $((\succ_u)_{u\in U}, (\succ_v)_{v\in V}, (\succ_{V,w}, \succ_{U,w})_{w\in W}) \in \mathcal{R}'$ for some linear orders $(\succ_v)_{v\in V}$ and $(\succ_{V,w})_{w\in W}$. Let $\Theta : \mathcal{S}(\mathcal{R}') \to \theta(\mathcal{M})$ and $\Psi : \mathcal{Q}(\mathcal{R}') \to \psi(\mathcal{M})$ be the mechanisms defined in the proof of Theorem 4. Hence, $\text{DA}_{\text{W},3} \equiv \Phi_{\Theta,\Psi}$ in the sub-domain $\mathcal{R}'$.

When $\mathcal{R}'$ is UW-acyclic-unrestricted, the result of Proposition 2 still holds.[15] Hence, analogous arguments to those made in the proof of Theorem 4 ensure that (i) and (ii) are equivalent.

Since the sub-domain $\mathcal{R}' \subseteq \mathcal{R}$ is UW-acyclic-unrestricted, Ergin (2002, Theorem 1) guarantees that the following properties hold: (i) Θ is Pareto efficient for V; and (ii) Ψ is Pareto efficient for W if and only if $\mathcal{R}'$ if V-Ergin-acyclic (cf., Narita (2021)). Since Proposition 3 ensures that $\text{DA}_{\text{W},3}[(\succ_h)_{h\in H}]$ is Pareto efficient for W if and only if both $\Theta[(\succ_h)_{h\in H}]$ is Pareto efficient for V and $\Psi[(\succ_h)_{h\in H}]$ is Pareto efficient for W, it follows that (ii) and (iii) are equivalent. ☐

Sequentially applying *serial dictatorship algorithms*, we will show that there is a mechanism that is both Pareto efficient and strategy-proof for agents in $A \in \{U \cup V, W\}$. Notice that this result and Theorem 3 will imply that one-sided Pareto efficiency dominates stability at the moment of finding a mechanism that is strategy-proof for agents in W.

Theorem 6. *Given $A \in \{U \cup V, W\}$, there exists a mechanism defined on the domain of preferences $\mathcal{R}$ that is both Pareto efficient and strategy-proof for A.*

Proof. Let $n = |U| = |V| = |W|$ and $f : \{1, \ldots, n\} \to W$ be a bijective function and $\Omega_{\text{W},f} : \mathcal{R} \to \mathcal{M}$ be the mechanism that associates to any $(\succ_h)_{h\in H} \in \mathcal{R}$ the matching obtained by the following application of the serial dictatorship algorithm:

Step 1. The agent $f(1) \in W$ is assigned her top choice under $\succ_{f(1)}$, denoted by (u_1, v_1).
Step k. For every $k \in \{2, \ldots, n\}$, the agent $f(k) \in W$ is assigned her top choice under $\succ_{f(k)}$ among the remaining alternatives $(U \times V) \setminus \{(u_i, v_i)\}_{1\leq i\leq k-1}$, denoted by (u_k, v_k).
Define $\Omega_{\text{W},f}[(\succ_h)_{h\in H}] = \{(u_k, v_k, f(k)) : k \in \{1, \ldots, n\}\}$.

It follows from Sönmez and Ünver (2011, Theorem 1) that the mechanism $\Omega_{\text{W},f}$ is strategy-proof for W and Pareto efficient for W in the whole domain $\mathcal{R}$.

[15] In the proof of Proposition 2, the domain of linear orders of agents in U plays a role only in the step 3 of the proof that (i) implies (ii). It is not difficult to verify that the arguments made in that part of the proof continue to hold when linear orders $(\succ_u)_{u\in U}$ are restricted to be Ergin-acyclic.

Analogously, given bijective functions $f : \{1, \dots, n\} \to U$ and $g : \{1, \dots, n\} \to V$, consider the mechanism $\Omega_{\mathrm{UV},(f,g)} : \mathcal{R} \to \mathcal{M}$ that associates to any $(\succ_h)_{h \in H} \in \mathcal{R}$ the matching obtained by the following *sequential* application of the serial dictatorship algorithm:

Step 1. The agent $f(1) \in U$ is assigned her top choice under $\succ_{f(1)}$, denoted by v_1.
The agent $g(1) \in V$ is assigned her top choice under $\succ_{g(1)}$, denoted by w_1.

Step k. For every $k \in \{2, \dots, n\}$, the agent $f(k) \in U$ is assigned her top choice under $\succ_{f(k)}$ among the remaining alternatives $V \setminus \{v_i\}_{1 \leq i \leq k-1}$, denoted by v_k.
For every $k \in \{2, \dots, n\}$, the agent $g(k) \in V$ is assigned her top choice under $\succ_{g(k)}$ among the remaining alternatives $W \setminus \{w_i\}_{1 \leq i \leq k-1}$, denoted by w_k.

If $r : \{1, \dots, n\} \to \{1, \dots, n\}$ is the bijective function implicitly defined by $g(r(k)) = v_k, \ \forall k \in \{1, \dots, n\}$, define $\Omega_{\mathrm{UV},(f,g)}[(\succ_h)_{h \in H}] = \{(f(k), v_k, w_{r(k)}) : k \in \{1, \dots, n\}\}$.

Since agents in $U \cup V$ only consider one side of the market in their preferences, the mechanism $\Omega_{\mathrm{UV},(f,g)}$ is both strategy-proof and Pareto efficient for them (cf., Sönmez and Ünver (2011)). $\square$

Given a preference profile $(\succ_h)_{h \in H} \in \mathcal{R}$, for any side of the market it is possible to determine the set of Pareto efficient allocations using the mechanisms described in the proof of Theorem 6. More formally, if we denote by $\mathbb{P}[A, (\succ_h)_{h \in H}] \subseteq \mathcal{M}$ the set of Pareto efficient allocations for agents of $A \subseteq H$ under $(\succ_h)_{h \in H}$, then the following properties hold:

$$\begin{aligned} \mathbb{P}[U \cup V, (\succ_h)_{h \in H}] &= \bigcup_{(f,g) \in \mathcal{B}(U) \times \mathcal{B}(V)} \Omega_{\mathrm{UV},(f,g)}[(\succ_h)_{h \in H}], \\ \mathbb{P}[W, (\succ_h)_{h \in H}] &= \bigcup_{f \in \mathcal{B}(W)} \Omega_{\mathrm{W},f}[(\succ_h)_{h \in H}], \end{aligned}$$

where $\mathcal{B}(A)$ is the set of bijective functions $f : \{1, \dots, |A|\} \to A$ (see Proposition A2 in the Appendix). Abdulkadiroğlu and Sönmez (1998) show that a similar property holds in two-sided markets.

7. Concluding remarks

In this paper, we studied stability, efficiency, and incentives in a class of solvable three-sided matching problems. The sets of stable and Pareto efficient outcomes were characterized and it was shown that there is a stable mechanism that is strategy-proof for the agents that ignore the trilateral structure of the market. Moreover, independently of the side of the market, it was possible to find a mechanism that is Pareto efficient and strategy-proof for its members.

However, we showed that no stable mechanism is strategy-proof for those agents that internalize the trilateral structure of the market in their preferences. Since this incompatibility between stability and strategy-proofness can be overcome only under strong restrictions on preferences, one-sided efficiency dominates stability when the focus is on mechanisms that are strategy-proof for W.

The structure of agents' preferences was crucial to show many of these results. It allowed us to characterize stability, efficiency, and strategy-proofness in terms of related properties of marriage markets. Hence, we were able to apply some classical results of the two-sided matching theory to characterize matchings and mechanisms in our framework.

To the best of our knowledge, the study of incentives in multi-sided matching problems is still incipient (cf., Block, Cantala, and Gibaja (2020, 2022)). For this reason, it is natural to ask if our results hold in other classes of three-sided matching problems.[16] Moreover, it is also interesting to analyze the incentives of agents to reveal information in multi-sided matching problems (cf., Sherstyuk (1999), Ostrovsky (2008), Hofbauer (2016)). The study of these topics is a matter for future research.

Appendix

Proof of Proposition 1. If $M \in \mathcal{M}$ is a stable matching of $[U, V, W, (\succ_h)_{h\in H}]$, then the following arguments guarantee that properties (i) and (ii) hold:

- If $\theta(M)$ is unstable in $(U, V, (\succ_u)_{u\in U}, (\succ_{U,M(v)})_{v\in V})$, then there is $(v, u) \in V \times U$ such that $v \succ_u \theta(M)(u)$ and $u \succ_{U,M(v)} \theta(M)(v)$.[17] It follows that $(u, v, M(v))$ blocks M. Indeed, when the triplet $(u, v, M(v))$ is formed, the situation of agent v does not change, $v \succ_u M(u)$, and $(v, u) \succ_{M(v)} M(M(v))$. This contradicts the stability of M.
- If $\psi(M)$ is unstable in $(V, W, (\succ_v)_{v\in V}, (\succ_{V,w})_{w\in W})$, then there is $(v, w) \in V \times W$ such that $w \succ_v \psi(M)(v)$ and $v \succ_{V,w} \psi(M)(w)$. Let $u \in U$ such that $M(u) = v$. Then, the triplet (u, v, w) blocks M, because the situation of u does not change, $w \succ_v M(v)$, and $(v, u) \succ_w M(w)$. This contradicts the stability of the matching M.

On the other hand, it follows from the proof of Theorem 1 that $M \in \mathcal{M}$ is stable in the three-sided problem $[U, V, W, (\succ_h)_{h\in H}]$ as long as (i) and (ii) hold. □

Remember that, for every $M \in \mathcal{M}$ we define

$$\begin{aligned}\theta(M) &= \{(u, v) \in U \times V : M(u) = v\},\\ \psi(M) &= \{(v, w) \in V \times W : M(v) = w\}.\end{aligned}$$

Moreover, given $\Theta : \mathcal{S} \to \theta(\mathcal{M})$ and $\Psi : \mathcal{Q} \to \psi(\mathcal{M})$, the mechanism $\Phi_{\Theta,\Psi} : \mathcal{R} \to \mathcal{M}$ associates with each $(\succ_h)_{h\in H} \in \mathcal{R}$ the set of triplets (u, v, w) such that

$$\Theta[(\succ_u)_{u\in U}, (\succ_{U,\overline{N}(v)})_{v\in V}](u) = v = \Psi[(\succ_v)_{v\in V}, (\succ_{V,w})_{w\in W}](w),$$

where $\overline{N} = \Psi[(\succ_v)_{v\in V}, (\succ_{V,w})_{w\in W}]$. Therefore, the following properties are satisfied

$$\theta(\Phi_{\Theta,\Psi}[(\succ_h)_{h\in H}]) = \Theta[(\succ_u)_{u\in U}, (\succ_{U,\overline{N}(v)})_{v\in V}], \qquad \psi(\Phi_{\Theta,\Psi}[(\succ_h)_{h\in H}]) = \Psi[(\succ_v)_{v\in V}, (\succ_{V,w})_{w\in W}].$$

Proof of Proposition 2. The fact that (i) implies (ii) is a consequence of the following three steps:

Step 1. *If $\Phi_{\Theta,\Psi} : \mathcal{R}' \to \mathcal{M}$ is strategy-proof for W, then $\Theta : \mathcal{S} \to \theta(\mathcal{M})$ is strategy-proof for V.*

Suppose that $\Theta : \mathcal{S} \to \theta(\mathcal{M})$ is not strategy-proof for V. Hence, there is an agent $\tilde{v} \in V$, a preference profile $S = (S_h)_{h\in U\cup V} \in \mathcal{S}$, and some linear order $S'_{\tilde{v}}$ defined on U such that $\Theta[S_{-\tilde{v}}, S'_{\tilde{v}}](\tilde{v})\ S_{\tilde{v}}\ \Theta[S](\tilde{v})$.

Since $\mathcal{R}' \subseteq \mathcal{R}$ is a UW-unrestricted sub-domain, we can consider any preference profile $(\succ_h)_{h\in H} \in \mathcal{R}'$ which complies with following conditions:

[16]Evidently, one way to adapt our findings to other frameworks is to ensure that the relationships between three-sided and two-sided markets remain valid (see Propositions 1, 2, and 3).

[17]Given $M \in \mathcal{M}$ and $(u, v, w) \in M$, we denote $\theta(M)(u) = v$, $\theta(M)(v) = u$, $\psi(M)(v) = w$, and $\psi(M)(w) = v$.

- For each $u \in U$, $\succ_u = S_u$.
- For each $v \in V$, $\succ_v$ is an arbitrary linear order defined on W.
- For each $w \in W$, $\succ_w$ is represented by $(\succ_{V,w}, \succ_{U,w})$, where $\succ_{V,w}$ is an arbitrary linear order defined on V, $\succ_{U,w} = S_{N(w)}$, and $N = \Psi[(\succ_v)_{v\in V}, (\succ_{V,w})_{w\in W}]$.

Let $\tilde{w} = N(\tilde{v})$ and $\succ'_{\tilde{w}}$ be a VU-lexicographic linear order represented by $(\succ_{V,\tilde{w}}, S'_{\tilde{v}})$.

Then, the property $\Theta[S_{-\tilde{v}}, S'_{\tilde{v}}](\tilde{v}) \; S_{\tilde{v}} \; \Theta[S](\tilde{v})$ can be rewritten as

$$\theta(\Phi_{\Theta,\Psi}[(\succ_h)_{h\neq\tilde{w}}, \succ'_{\tilde{w}}])(\tilde{v}) \; \succ_{U,\tilde{w}} \; \theta(\Phi_{\Theta,\Psi}[(\succ_h)_{h\in H}])(\tilde{v}).$$

Notice that, the definition of $\Phi_{\Theta,\Psi}$ ensures that

$$\psi(\Phi_{\Theta,\Psi}[(\succ_h)_{h\neq\tilde{w}}, \succ'_{\tilde{w}}])(\tilde{w}) = \psi(\Phi_{\Theta,\Psi}[(\succ_h)_{h\in H}])(\tilde{w}).$$

Since $\tilde{v} = N(\tilde{w})$ and $\succ_{\tilde{w}}$ is VU-lexicographic, it follows that

$$\Phi_{\Theta,\Psi}[(\succ_h)_{h\neq\tilde{w}}, \succ'_{\tilde{w}}](\tilde{w}) \; \succ_{\tilde{w}} \; \Phi_{\Theta,\Psi}[(\succ_h)_{h\in H}](\tilde{w}).$$

Thus, the mechanism $\Phi_{\Theta,\Psi}$ is not strategy-proof for W. *Q.E.D.*

Step 2. *If $\Phi_{\Theta,\Psi} : \mathcal{R}' \to \mathcal{M}$ is strategy-proof for W, then $\Psi : \mathcal{Q}(\mathcal{R}') \to \psi(\mathcal{M})$ is strategy-proof for W.*

Suppose that $\Psi : \mathcal{Q}(\mathcal{R}') \to \psi(\mathcal{M})$ is not strategy-proof for W. Hence, there is an agent $\tilde{w} \in W$, a preference profile $Q = (Q_h)_{h\in V\cup W} \in \mathcal{Q}(\mathcal{R}')$, and some linear order $Q'_{\tilde{w}}$ defined on V such that

$$(Q_{-\tilde{w}}, Q'_{\tilde{w}}) \in \mathcal{Q}(\mathcal{R}') \qquad \text{and} \qquad \Psi[Q_{-\tilde{w}}, Q'_{\tilde{w}}](\tilde{w}) \; Q_{\tilde{w}} \; \Psi[Q](\tilde{w}).$$

By the definition of $\mathcal{Q}(\mathcal{R}')$, we can consider any preference profile $(\succ_h)_{h\in H} \in \mathcal{R}'$ satisfying the following conditions:

- For each $u \in U$, $\succ_u$ is an arbitrary linear order defined on V.
- For each $v \in V$, $\succ_v = Q_v$.
- For each $w \in W$, $\succ_w$ is represented by $(\succ_{V,w}, \succ_{U,w})$, where $\succ_{V,w} = Q_w$ and $\succ_{U,w}$ is an arbitrary linear order defined on U.

Let $\succ'_{\tilde{w}}$ be a VU-lexicographic linear order represented by $(Q'_{\tilde{w}}, \succ'_{U,\tilde{w}})$, where $\succ'_{U,\tilde{w}}$ is an arbitrary linear order defined on U. Then, the property $\Psi[Q_{-\tilde{w}}, Q'_{\tilde{w}}](\tilde{w}) \; Q_{\tilde{w}} \; \Psi[Q](\tilde{w})$ can be rewritten as

$$\psi(\Phi_{\Theta,\Psi}[(\succ_h)_{h\neq\tilde{w}}, \succ'_{\tilde{w}}])(\tilde{w}) \; \succ_{V,\tilde{w}} \; \psi(\Phi_{\Theta,\Psi}[(\succ_h)_{h\in H}])(\tilde{w}).$$

Since $\succ_{\tilde{w}}$ is VU-lexicographic, it follows from the definition of ψ that

$$\Phi_{\Theta,\Psi}[(\succ_h)_{h\neq\tilde{w}}, \succ'_{\tilde{w}}](\tilde{w}) \; \succ_{\tilde{w}} \; \Phi_{\Theta,\Psi}[(\succ_h)_{h\in H}](\tilde{w}).$$

Thus, the mechanism $\Phi_{\Theta,\Psi}$ is not strategy-proof for W. *Q.E.D.*

Step 3. *If $\Phi_{\Theta,\Psi} : \mathcal{R}' \to \mathcal{M}$ is strategy-proof for W, then $\Psi : \mathcal{Q}(\mathcal{R}') \to \psi(\mathcal{M})$ is non-bossy for W.*

Suppose that $\Psi : \mathcal{Q}(\mathcal{R}') \to \psi(\mathcal{M})$ is bossy for W. Hence, there is an agent $\tilde{w} \in W$, a preference profile $Q = (Q_h)_{h\in V\cup W} \in \mathcal{Q}(\mathcal{R}')$, and some linear order $Q'_{\tilde{w}}$ defined on V such that $(Q_{-\tilde{w}}, Q'_{\tilde{w}}) \in \mathcal{Q}(\mathcal{R}')$,

$$\tilde{v} \equiv \Psi[Q_{-\tilde{w}}, Q'_{\tilde{w}}](\tilde{w}) \; = \; \Psi[Q](\tilde{w}) \qquad \wedge \qquad \Psi[Q_{-\tilde{w}}, Q'_{\tilde{w}}] \; \neq \; \Psi[Q].$$

Since $|V| = |W|$, there are agents $v_1 \in V$ and $w_1, w_2 \in W \setminus \{\tilde{w}\}$ such that

$$w_1 \equiv \Psi[Q](v_1) \; \neq \; \Psi[Q_{-\tilde{w}}, Q'_{\tilde{w}}](v_1) \equiv w_2.$$

Since $\mathcal{R}' \subseteq \mathcal{R}$ is a UW-unrestricted sub-domain, when $U = \{u_1, \ldots, u_{n-1}, \tilde{u}\}$, $V = \{v_1, \ldots, v_{n-1}, \tilde{v}\}$ and $W = \{w_1, \ldots, w_{n-1}, \tilde{w}\}$, it follows from the definition of $\mathcal{Q}(\mathcal{R}')$ that we can consider any preference profile $(\succ_h)_{h \in H} \in \mathcal{R}'$ satisfying the following conditions:

- The linear orders $\succ_{u_1}$, $\succ_{u_2}$, and $\succ_{\tilde{u}}$ are such that

$\succ_{u_1}$	$\succ_{u_2}$	$\succ_{\tilde{u}}$
v_1	v_1	$\tilde{v}$
$\tilde{v}$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$

- For $u \in U \setminus \{u_1, u_2, \tilde{u}\}$, $\succ_u$ is an arbitrary linear order defined on V.
- For each $v \in V$, $\succ_v = Q_v$.
- For each $w \in W$ there is a linear order $\succ_{U,w}$ defined on U such that $\succ_w$ is represented by $(Q_w, \succ_{U,w})$, where $\succ_{U,w_1}$, $\succ_{U,w_2}$, and $\succ_{U,\tilde{w}}$ are such that

$\succ_{U,w_1}$	$\succ_{U,w_2}$	$\succ_{U,\tilde{w}}$
u_1	u_2	u_1
$\vdots$	$\vdots$	$\tilde{u}$
$\vdots$	$\vdots$	$\vdots$

Let $\succ'_{\tilde{w}}$ be the VU-lexicographic linear order represented by $(Q'_{\tilde{w}}, \succ_{U,\tilde{w}})$. It follows that the properties $\Psi[Q_{-\tilde{w}}, Q'_{\tilde{w}}](\tilde{w}) = \Psi[Q](\tilde{w})$ and $\Psi[Q_{-\tilde{w}}, Q'_{\tilde{w}}] \neq \Psi[Q]$ can be rewritten as

$$\begin{aligned} \psi(\Phi_{\Theta,\Psi}[(\succ_h)_{h \neq \tilde{w}}, \succ'_{\tilde{w}}])(\tilde{w}) &= \psi(\Phi_{\Theta,\Psi}[(\succ_h)_{h \in H}])(\tilde{w}), \\ \psi(\Phi_{\Theta,\Psi}[(\succ_h)_{h \neq \tilde{w}}, \succ'_{\tilde{w}}]) &\neq \psi(\Phi_{\Theta,\Psi}[(\succ_h)_{h \in H}]). \end{aligned}$$

Let $M = \Phi_{\Theta,\Psi}[(\succ_h)_{h \in H}]$ and $M' = \Phi_{\Theta,\Psi}[(\succ_h)_{h \neq \tilde{w}}, \succ'_{\tilde{w}}]$.

Since $M(v_1) = w_1$ and $M'(v_1) = w_2$, the definitions of $(\succ_h)_{h \in H}$ and $\succ'_{\tilde{w}}$ guarantee that agents $\tilde{v}$ and $\tilde{u}$ form a couple in any stable matching of the marriage market $(U, V, (\succ_u)_{u \in U}, (\succ_{U,M(v)})_{v \in V})$. Analogously, agents $\tilde{v}$ and u_1 form a couple in any stable matching of $(U, V, (\succ_u)_{u \in U}, (\succ_{U,M'(v)})_{v \in V})$.

Therefore, it follows from Proposition 1 that

$$\Phi_{\Theta,\Psi}[(\succ_h)_{h \in H}](\tilde{w}) = (\tilde{v}, \tilde{u}) \qquad \wedge \qquad \Phi_{\Theta,\Psi}[(\succ_h)_{h \neq \tilde{w}}, \succ'_{\tilde{w}}](\tilde{w}) = (\tilde{v}, u_1).$$

We conclude that

$$\Phi_{\Theta,\Psi}[(\succ_h)_{h \neq \tilde{w}}, \succ'_{\tilde{w}}](\tilde{w}) \succ_{\tilde{w}} \Phi_{\Theta,\Psi}[(\succ_h)_{h \in H}](\tilde{w}),$$

which implies that $\Phi_{\Theta,\Psi}$ is not strategy-proof for W. *Q.E.D.*

In order to show that (ii) implies (i), suppose that the mechanism $\Theta : \mathcal{S} \to \theta(\mathcal{M})$ is strategy-proof for V and the mechanism $\Psi : \mathcal{Q}(\mathcal{R}') \to \psi(\mathcal{M})$ is strategy-proof and non-bossy for W. By contradiction, assume that $\Phi_{\Theta,\Psi} : \mathcal{R}' \to \mathcal{M}$ is not strategy-proof for W. Hence, there is an agent $\tilde{w} \in W$, a profile $(\succ_h)_{h \in H} \in \mathcal{R}'$ and linear order $\succ'_{\tilde{w}}$ such that $((\succ_h)_{h \neq \tilde{w}}, \succ'_{\tilde{w}}) \in \mathcal{R}'$ and

$$(v', u') = \Phi_{\Theta,\Psi}[(\succ_h)_{h \neq \tilde{w}}, \succ'_{\tilde{w}}](\tilde{w}) \succ_{\tilde{w}} \Phi_{\Theta,\Psi}[(\succ_h)_{h \in H}](\tilde{w}) = (\tilde{v}, \tilde{u}).$$

If $(\succ_{V,w}, \succ_{U,w})$ are the linear orders representing $\succ_w$, consider the profile $Q = (Q_h)_{h \in V \cup W} \in \mathcal{Q}(\mathcal{R}')$ such that $Q_v = \succ_v$ for all $v \in V$, $Q_w = \succ_{V,w}$ for all $w \in W$. Also, if $(\succ'_{V,\tilde{w}}, \succ'_{U,\tilde{w}})$ are the linear orders representing $\succ'_{\tilde{w}}$, let $Q'_{\tilde{w}}$ be a linear order defined on V such that $Q'_{\tilde{w}} = \succ'_{V,\tilde{w}}$.

Suppose that $v' \neq \tilde{v}$. Since $\succ_{\tilde{w}}$ is VU-lexicographic, it follows that $v' \succ_{V,\tilde{w}} \tilde{v}$. Moreover, by the definition of $\Phi_{\Theta,\Psi}$, we have that $\succ'_{V,\tilde{w}} \neq \succ_{V,\tilde{w}}$. Since the property $v' \succ_{V,\tilde{w}} \tilde{v}$ is equivalent to

$$v' = \psi(\Phi_{\Theta,\Psi}[(\succ_h)_{h\neq\tilde{w}}, \succ'_{\tilde{w}}])(\tilde{w}) \succ_{V,\tilde{w}} \psi(\Phi_{\Theta,\Psi}[(\succ_h)_{h\in H}])(\tilde{w}) = \tilde{v},$$

by the definition of ψ we have that

$$\Psi[Q_{-\tilde{w}}, Q'_{\tilde{w}}](\tilde{w})\, Q_{\tilde{w}}\, \Psi[Q](\tilde{w}).$$

That is, Ψ is not strategy-proof for W. A contradiction.

Suppose that $v' = \tilde{v}$ *and* $u' \neq \tilde{u}$. Since $\succ_{\tilde{w}}$ is VU-lexicographic, it follows that $u' \succ_{U,\tilde{w}} \tilde{u}$. Also, there are two possibilities depending on the characteristics of the linear orders $(\succ'_{V,\tilde{w}}, \succ'_{U,\tilde{w}})$ representing $\succ'_{\tilde{w}}$:

(1) *Suppose that* $\succ'_{V,\tilde{w}} \neq \succ_{V,\tilde{w}}$ *and* $\succ'_{U,\tilde{w}} = \succ_{U,\tilde{w}}$. Since $u' \neq \tilde{u}$ and $\Psi[Q_{-\tilde{w}}, Q'_{\tilde{w}}](\tilde{w}) = \Psi[Q](\tilde{w})$, it follows from the definition of $\Phi_{\Theta,\Psi}$ that there is an agent $\hat{w} \in W$ such that $\Psi[Q_{-\tilde{w}}, Q'_{\tilde{w}}](\hat{w}) \neq \Psi[Q](\hat{w})$. Thus, the mechanism Ψ is bossy for W. A contradiction.

(2) *Suppose that* $\succ'_{U,\tilde{w}} \neq \succ_{U,\tilde{w}}$. If $M = \Phi_{\Theta,\Psi}[(\succ_h)_{h\in H}]$, let $S = (S_h)_{h\in U\cup V} \in \mathcal{S}$ be such that $S_u = \succ_u$ for all $u \in U$, $S_v = \succ_{U,M(v)}$ for all $v \in V$, and $S'_{\tilde{v}} = \succ'_{U,M(\tilde{v})}$. Since $u' \succ_{U,\tilde{w}} \tilde{u}$ can be rewritten as

$$u' = \theta(\Phi_{\Theta,\Psi}[(\succ_h)_{h\neq\tilde{w}}, \succ'_{\tilde{w}}])(\tilde{v}) \succ_{U,\tilde{w}} \theta(\Phi_{\Theta,\Psi}[(\succ_h)_{h\in H}])(\tilde{v}) = \tilde{u},$$

it follows from the definition of θ that

$$\Theta[S_{-\tilde{v}}, S'_{\tilde{v}}](\tilde{v})\, S_{\tilde{v}}\, \Theta[S](\tilde{v}).$$

That is, Θ is not strategy-proof for V. A contradiction. □

Proof of Proposition 3. If $M \in \mathcal{M}$ is Pareto efficient for W in $[U, V, W, (\succ_h)_{h\in H}]$, then the following arguments guarantee that properties (i) and (ii) hold:

- If $\theta(M)$ is Pareto inefficient for V in the marriage market $(U, V, (\succ_u)_{u\in U}, (\succ_{U,M(v)})_{v\in V})$, then there exists a matching $N \in \mathcal{M}$ such that
  - For any $v \in V$, either $\theta(N)(v) \succ_{U,M(v)} \theta(M)(v)$ or $N(v) = \theta(M)(v)$.
  - For some $v \in V$ we have that $\theta(N)(v) \succ_{U,M(v)} \theta(M)(v)$.

  Since $(\succ_w)_{w\in W}$ are represented by the linear orders $(\succ_{V,w}, \succ_{U,w})_{w\in W}$, for agents in W the matching $\{(\theta(N)(v), v, M(v)) : v \in V\}$ Pareto dominates M. A contradiction.
- If $\psi(M)$ is Pareto inefficient for W in the marriage market $(V, W, (\succ_v)_{v\in V}, (\succ_{V,w})_{w\in W})$, then there exists a matching $N \in \mathcal{M}$ such that
  - For any $w \in W$, either $\psi(N)(w) \succ_{V,w} \psi(M)(w)$ or $\psi(N)(w) = \psi(M)(w)$.
  - For some $w \in W$ we have that $\psi(N)(w) \succ_{V,w} \psi(M)(w)$.

  Hence, for agents in W the matching M is Pareto dominated by N. A contradiction.

On the other hand, suppose that properties (i) and (ii) hold for some $M \in \mathcal{M}$. If the matching M is Pareto inefficient for W, then there exists $N \in \mathcal{M}$ such that

- For any $w \in W$, either $N(w) \succ_w M(w)$ or $N(w) = M(w)$.
- For some $w \in W$ we have that $N(w) \succ_w M(w)$.

Suppose that there exists $\overline{v} \in V$ such that $N(\overline{v}) \neq M(\overline{v})$. Since $(\succ_w)_{w\in W}$ are strict VU-lexicographic preferences, if $\overline{w} = M(\overline{v})$ and $N(\overline{w}) = (\tilde{v}, \tilde{u})$, it follows that $\overline{v} \neq \tilde{v}$ and $\tilde{v} \succ_{V,\overline{w}} \overline{v}$. Notice that, for every $w \in W$, $N(w) \succ_w M(w)$ implies that either $\psi(N)(w) \succ_{V,w} \psi(M)(w)$ or $\psi(N)(w) = \psi(M)(w)$. We conclude that $\psi(M)$ is Pareto inefficient for W in $(V, W, (\succ_v)_{v\in V}, (\succ_{V,w})_{w\in W})$. A contradiction.

Alternatively, suppose that $N(v) = M(v)$ for all $v \in V$. Given $v \in V$ such that $w = M(v)$, it follows that $N(w) \succ_w M(w)$ implies that $\theta(N)(v) \succ_{U,M(v)} \theta(M)(v)$. Moreover, there is $\overline{u} \in U$ such that $N(\overline{u}) \neq M(\overline{u})$. If $(\overline{u}, \overline{v}, \overline{w}) \in M$ and $N(\overline{w}) = (\overline{v}, \tilde{u})$, it follows that $\overline{u} \neq \tilde{u}$ and $\tilde{u} \succ_{U,\overline{w}} \overline{u}$. We conclude that $\theta(M)$ is Pareto inefficient for V in $(U, V, (\succ_u)_{u\in U}, (\succ_{U,M(v)})_{v\in V})$. A contradiction. □

Proposition A1. *There is no $\Psi : \mathcal{Q} \to \psi(\mathcal{M})$ stable, strategy-proof for W, and non-bossy for W.*

Proof. Let $[V, W, (Q_h)_{h\in V\cup W}]$ be such that $V = \{v_1, v_2, v_3, v_4\}$, $W = \{w_1, w_2, w_3, w_4\}$, and

Q_{v_1}	Q_{v_2}	Q_{v_3}	Q_{v_4}
w_2	w_1	w_3	w_2
w_3	w_2	$\vdots$	w_4
w_1	$\vdots$	$\vdots$	$\vdots$
w_4	$\vdots$	$\vdots$	$\vdots$

Q_{w_1}	Q_{w_2}	Q_{w_3}	Q_{w_4}
v_1	v_2	v_3	v_4
v_2	v_1	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

In this context, the only stable matchings are

$$\begin{aligned} \mu &= \{(v_1, w_2), (v_2, w_1), (v_3, w_3), (v_4, w_4)\}, \\ \mu' &= \{(v_1, w_1), (v_2, w_2), (v_3, w_3), (v_4, w_4)\}. \end{aligned}$$

Therefore, given a stable mechanism $\Psi : \mathcal{Q} \to \psi(\mathcal{M})$, we have two possibilities:

(i) $\Psi[(Q_h)_{h\in V\cup W}] = \mu$. In this case, if agents $h \neq w_2$ report their true preferences, then w_2 improves her situation by reporting $v_2 Q^*_{w_2} v_4 Q^*_{w_2} v_3 Q^*_{w_2} v_1$ instead of Q_{w_2}. Indeed, μ' is the only stable matching in this scenario and $v_2 Q_{w_2} v_1$. Thus, Ψ is not strategy-proof for W in the domain $\mathcal{Q}$.

(ii) $\Psi[(Q_h)_{h\in V\cup W}] = \mu'$. In this case, if agents $h \neq w_3$ report their true preferences and w_3 reports $v_1 Q^*_{w_3} v_3 Q^*_{w_3} v_2 Q^*_{w_3} v_4$ instead of Q_{w_3}, the only matching stable is μ. Since $\mu(w_3) = \mu'(w_3)$ and $\mu \neq \mu'$, it follows that Ψ is bossy for W in the domain $\mathcal{Q}$.

We conclude that Ψ cannot be at the same time strategy-proof and non-bossy for W. □

Proposition A2. *Given a preference profile $(\succ_h)_{h\in H} \in \mathcal{R}$, we have that*

$$\begin{aligned} \mathbb{P}[U \cup V, (\succ_h)_{h\in H}] &= \bigcup_{(f,g)\in\mathcal{B}(U)\times\mathcal{B}(V)} \Omega_{\mathrm{UV},(f,g)}[(\succ_h)_{h\in H}], \\ \mathbb{P}[W, (\succ_h)_{h\in H}] &= \bigcup_{f\in\mathcal{B}(W)} \Omega_{\mathrm{W},f}[(\succ_h)_{h\in H}], \end{aligned}$$

where $\mathcal{B}(A)$ is the set of bijective functions $f : \{1, \ldots, |A|\} \to A$.

Proof. Let $n = |U| = |V| = |W|$. Given $(\succ_h)_{h\in H} \in \mathcal{R}$ and a matching M that is Pareto efficient for $U \cup V$ under $(\succ_h)_{h\in H} \in \mathcal{R}$, there are agents $u_1 \in U$ and $v_1 \in V$ whose relevant partners are their best alternatives in V and W, respectively. That is, $M(u_1) \succ_{u_1} v$ for all $v \in V \setminus \{M(u_1)\}$ and $M(v_1) \succ_{v_1} w$ for all $w \in W \setminus \{M(v_1)\}$. Analogously, for any $k \in \{2, \ldots, n\}$, there exist agents $u_k \in U$ and $v_k \in V$ whose relevant partners are their best alternatives in $V \setminus \{M(u_1), \ldots, M(u_{k-1})\}$ and $W \setminus \{M(v_1), \ldots, M(w_{k-1})\}$, respectively. If the functions $\overline{f} \in \mathcal{B}(U)$ and $\overline{g} \in \mathcal{B}(V)$ are such that $(\overline{f}(i), \overline{g}(i)) = (u_i, v_i)$ for all $i \in \{1, \ldots, n\}$, then $M = \Omega_{\mathrm{UV},(\overline{f},\overline{g})}[(\succ_h)_{h\in H}]$. We conclude that

$$\mathbb{P}[U \cup V, (\succ_h)_{h\in H}] \subseteq \bigcup_{(f,g)\in\mathcal{B}(U)\times\mathcal{B}(V)} \Omega_{\mathrm{UV},(f,g)}[(\succ_h)_{h\in H}].$$

On the other side, Sönmez and Ünver (2011, Theorem 1) ensure that

$$\bigcup_{(f,g)\in\mathcal{B}(U)\times\mathcal{B}(V)} \Omega_{\mathrm{UV},(f,g)}[(\succ_h)_{h\in H}] \subseteq \mathbb{P}[U\cup V,(\succ_h)_{h\in H}].$$

The equivalence between $\mathbb{P}[W,(\succ_h)_{h\in H}]$ and $\bigcup_{f\in\mathcal{B}(W)} \Omega_{\mathrm{W},f}[(\succ_h)_{h\in H}]$ follows by similar arguments. □

References

[1] Abdulkadiroğlu, A., and T. Sönmez (1998): "Random serial dictatorship and the core from random endowments in house allocation problems," *Econometrica*, 66, 689-701.

[2] Alcalde, J., and S. Barberà (1994): "Top dominance and the possibility of strategy-proof stable solutions to matching problems," *Economic Theory*, 4, 417-435.

[3] Alkan, A. (1988): "Non-existence of stable threesome matchings," *Mathematical Social Sciences*, 16, 207-209.

[4] Arenas, J., and J.P. Torres-Martínez (2022): "Reconsidering the existence of stable solutions in three-sided matching problems with mixed preferences," working paper. Available at http://dx.doi.org/10.2139/ssrn.4122308

[5] Bloch, F., D. Cantala, D. Gibaja (2020): "Matching through institutions," *Games and Economic Behavior*, 121, 204-231.

[6] Bloch, F., D. Cantala, D. Gibaja (2022): "The chair allocation problem," working paper. Available at https://www.sites.google.com/site/francisbloch1

[7] Boros, E., V. Gurvich, S. Jaslar, and D. Krasner (2004): "Stable matchings in three-sided systems with cyclic preferences," *Discrete Mathematics*, 289, 1-10.

[8] Biró, P., and E. McDermid (2010): "Three-sided stable matchings with cyclic preferences," *Algorithmica,* 58, 5-18.

[9] Danilov, V.I. (2003): "Existence of stable matchings in some three-sided systems," *Mathematical Social Sciences,* 46, 145-148.

[10] Dubins, L.E., and D.A. Freedman (1981) "Machiavelli and the Gale-Shapley algorithm," *American Mathematical Monthly*, 88, 485-494.

[11] Ergin, H. (2002): "Efficient resource allocation on the basis of priorities," *Econometrica*, 70, 2489-2497.

[12] Eriksson, K., J. Sjöstrand, and P. Strimling (2006): "Three-dimensional stable matching with cyclic preferences," *Mathematical Social Sciences*, 52, 77-87.

[13] Gale, D., and L. Shapley (1962): "College admissions and the stability of marriage," *The American Mathematical Monthly,* 69, 9-15.

[14] Gale, D., and M. Sotomayor (1985): "Some remarks on the stable matching problem," *Discrete Applied Mathematics*, 11, 223-232.

[15] Hofbauer, J. (2016): "*d*-dimensional stable matching with cyclic preferences," *Mathematical Social Sciences*, 82, 72-76.

[16] Huang, C. (2010): "Circular stable matching and 3-way kidney transplant," *Algorithmica*, 58, 137-150.

[17] Knuth, D. (1976): "Mariages stables et leurs relations avec d'autres problèmes combinatoires," Les Presses de l'Université de Montréal, Montréal

[18] Lahiri, S. (2009) "Three-sided matchings and separable preferences," *Contributions to Game Theory and Management*, 2, 251-259.

[19] Lam, C-K, and C.G. Plaxton (2019): "On the existence of three-dimensional stable matchings with cyclic preferences," In: *Algorithmic Game Theory*, Lecture Notes in Computer Science, 11801, 329-342.

[20] Lam, C-K, and C.G. Plaxton (2021): "On the existence of three-dimensional stable matchings with cyclic preferences," *Theory of Computing Systems.* Available at https://doi.org/10.1007/s00224-021-10055-8

[21] Lerner, E.Y. (2022): "Instances of small size with no weakly stable matching for three-sided problem with complete cyclic preferences", working paper. Available at https://doi.org/10.48550/arXiv.2107.10102

[22] Narita, Y. (2021): "Comment on Efficient resource allocation on the basis of priorities," *Econometrica*, 89, 15-17.

[23] Ostrovsky, M. (2008): "Stability in supply chain networks," *American Economic Review*, 98, 897-923.

[24] Pápai, S. (2000): "Strategyproof assignment by hierarchical exchange," *Econometrica*, 68, 1403-1433.

[25] Pashkovich, K., and L. Poirrier (2020): "Three-dimensional stable matching with cyclic preferences," *Optimization Letters*, 14, 2615-2623.

[26] Raghavan, M. (2021): "Matching firms and workers through intermediaries," working paper. Available at http://dx.doi.org/10.2139/ssrn.3882447

[27] Roth, A. (1982): "The economics of matching: stability and incentives," *Mathematics of Operations Research*, 7, 617-628.

[28] Roth, A., and M. Sotomayor (1989): "College admissions problem revisited," *Econometrica*, 57, 559-570.

[29] Sherstyuk, K. (1999): "Multisided matching games with complementarities," *International Journal of Game Theory*, 28, 489-509.

[30] Sönmez, T., and U. Ünver (2011): "Matching, allocation, and exchange of discrete resources," Handbook of Social Economics, 1, 781-852.

[31] Zhang, F., J. Li, J. Fan, H. Shen, J. Shen, and H. Yu (2019): "Three-dimensional stable matching with hybrid preferences," *Journal of Combinatorial Optimization*, 37, 330-336.

[32] Zhang, F., and L. Zhong (2021): "Three-sided matching problem with mixed preferences," *Journal of Combinatorial Optimization,* 42, 928-936.

[33] Zhong, L., and Y. Bai (2019): "Three-sided stable matching problem with two of them as cooperative partners," *Journal of Combinatorial Optimization*, 37, 286-292.