FACULTAD ECONOMÍA Y NEGOCIOS UNIVERSIDAD DE CHILE
DEPARTAMENTO DE ECONOMÍA

SERIE DE DOCUMENTOS DE TRABAJO

SDT 404

Walrasian equilibrium as limit of a competitive equilibrium without divisible goods

Autores:
Michael Florig
Jorge Rivera Cayupi

Santiago, Mayo de 2015

Walrasian equilibrium as limit of a competitive equilibrium without divisible goods *

Michael Florig[†] Jorge Rivera[‡]

June 10, 2015

Abstract

We study economies where all commodities are indivisible at the individual level, but perfectly divisible at the aggregate level of the economy. Under the survival assumption, we show that a competitive outcome in the discrete economy, called rationing equilibrium, converges to a Walras equilibrium of the limit economy when the level of indivisibility becomes small. If the survival assumption does not hold at the limit economy, then the rationing equilibrium converges to a hierarchic equilibrium.

Keywords: competitive equilibrium, indivisible goods, convergence.

JEL Classification: C62, D50, E40

1 Introduction

In this paper we investigate the asymptotic behaviour of competitive equilibria existing in economies when discrete consumption and production sets converge to convex sets, where all goods are perfectly divisible. Since a Walras equilibrium may fail to exist in the case of discrete consumption sets, we base our analysis on the framework proposed in Florig and Rivera [9], using a continuum of agents and discrete consumption and production sets, so that goods are indivisible at the individual level, but perfectly divisible at the aggregate level of the entire economy.[1] Using a parameter called "fiat money" –whose solely role is to facilitate the exchange among individuals– and considering a

*This work was supported by FONDECYT - Chile, Project N°. 1000766-2000, and ICM Sistemas Complejos de Ingeniería.
We thank Yves Blalasko and Carlos Herves-Beloso for helpful comments.
[†]Bad Homburg, Germany, michael@florig.net.
[‡]Departamento de Economía, Universidad de Chile, Diagonal Paraguay 257, Torre 26, Santiago, Chile, jrivera@econ.uchile.cl.
[1]See Bobzin [4] for a survey on indivisible goods.

regularized notion of demand, existence of a competitive equilibrium notion, called rationing equilibrium, can be established for these discrete economies. The set of rationing equilibria contains the set of Walras equilibria.

The nature of the limit of the equilibrium sequence will depend on the assumptions imposed on the limit economy. When the strong survival condition holds, then the limit of rationing equilibria will be a Walras equilibrium,[2] and therefore the indivisibility of goods becomes indeed irrelevant when it is small. The situation might be quite different when the initial endowment of resources of each consumer does not belong to the interior of the respective consumption set. In such case, the indivisibility of goods might matter, independently of how small it is. It may occur that not all consumers have access to all goods, i.e. a good may be so expensive that some consumers who do not own the expensive goods cannot buy a single unit by selling their entire initial endowment. When the goods become "more divisible", i.e. if the minimal unit per good decreases, then the equilibrium price may react such that the situation persists.

Following Gay [10], based on a generalized concept of price, several authors have proposed generalizations of the Walras equilibrium existing in the convex case even when the Walras equilibrium does not exist due to a failure of the strong survival assumption (Danilov and Sotskow [5], Marakulin [13], Mertens [14], Florig [6]).[3] Supported by several examples, Florig [6] proposes an interpretation of those generalized prices in terms of small indivisibilities. In the case of linear preferences, Florig [7] shows that a hierarchic equilibrium as proposed in [6] is the limit of standard competitive equilibria of economies with discrete consumption sets converging to the positive orthant.[4]

We will show that rationing equilibria converge to a hierarchic equilibrium when the strong survival does not hold in the limit economy. This result formalises the interpretation of hierarchic equilibria in terms of small indivisibilities given in Florig [6]. In the absence of the strong survival assumption we may thus be in a situation where indivisibilities matter, independently how small the minimal tradable units of goods are. Note that a rationing equilibrium (with a positive price of fiat money) is a Walras equilibrium, provided that the initial endowment in fiat money is dispersed (Florig and Rivera [9]). Therefore our result does not depend on the concept of rationing equilibrium.

This work is organized as follows. In Section 2 we introduce some mathematical concepts that we use throughout this paper. In Section 3 we describe the model of discrete economies that approximate a standard economy, hence introducing a convergence concept for economies. In Section 4 we present conditions ensuring that the limit of a sequence of rationing equilibria is a Walras equilibrium (Proposition 4.1). In Section 5 we consider a more general framework, without

[2]If local non-satiation of preferences does not hold the price of fiat money may however be positive at the limit.

[3]When the consumption set is the positive orthant, the set of goods L is partitioned into several classes $L_1, \ldots, L_k$ according to their value, with any quantity of L_r goods buying infinite amounts of less valuable $L_{r+1}, \ldots, L_k$ goods, buying other goods in L_r at standard prices, and finally any quantity of L_r goods cannot buy any positive quantity of more valuable $L_1, \ldots, L_{r-1}$ goods.

[4]As local satiation cannot hold in the case of discrete consumption, dividend equilibria are employed.

a strong survival condition on the limit economy. In that case the limit of a sequence of rationing equilibrium is shown to be a hierarchic equilibrium.

2 Basic notation and preliminary concepts

In the following, 0_m is the origin of $\mathbb{R}^m$ and x^{t} is the transpose of $x \in \mathbb{R}^m$, whose Euclidean norm is $\|x\|$. The inner product between $x, y \in \mathbb{R}^m$ is $x \cdot y = x^{\mathrm{t}} y$, and the open ball with center x and radius $\varepsilon > 0$ is $\mathbb{B}(x, \varepsilon)$. For a couple of sets $K_1, K_2 \subseteq \mathbb{R}^m$, $\xi \in \mathbb{R}$ and $p \in \mathbb{R}^m$, we denote $\xi K_1 = \{\xi x,\, x \in K_1\}$, $p \cdot K_1 = \{p \cdot x,\, x \in K_1\}$ and $K_1 \pm K_2 = \{x_1 \pm x_2, x_1 \in K_1,\, x_2 \in K_2\}$, while the set-difference between them is denoted $K_1 \setminus K_2$. Additionally, $\operatorname{cl} K_1$, $\operatorname{int} K_1$ and $\operatorname{conv} K_1$ denote, respectively, the closure, interior and the convex hull of K_1.

By denoting

$$\mathbb{N}_\infty = \{N \subseteq \mathbb{N} \,|\, \mathbb{N} \setminus N \text{ is finite}\} \quad \text{and} \quad \mathbb{N}_\infty^* = \{N \subset \mathbb{N} \,|\, N \text{ is infinite}\},$$

we recall the outer limit of a sequence of subsets $\{K_n\}_{n \in \mathbb{N}}$ of $\mathbb{R}^m$ is the subset

$$\limsup_{n \to \infty} K_n \quad = \quad \{x \in \mathbb{R}^m \,|\, \exists \mathrm{N} \in \mathbb{N}_\infty^*,\ \exists x_n \in K_n,\, n \in \mathrm{N}, \text{ with } x_n \to_{\mathrm{N}} x\}, \tag{1}$$

while the inner limit is the set

$$\liminf_{n \to \infty} K_n \quad = \quad \{x \in \mathbb{R}^m \,|\, \exists \mathrm{N} \in \mathbb{N}_\infty,\ \exists x_n \in K_n, n \in \mathrm{N},\ \text{with } x_n \to_{\mathrm{N}} x\}.$$

We say that the sequence converges in the sense of Kuratowski – Painlevé to $K \subseteq \mathbb{R}^m$ if

$$\limsup_{n \to \infty} K_n = \liminf_{n \to \infty} K_n = K,$$

and we denote $\lim\limits_{n \to \infty} K_n = K$.

Finally, given $\mathrm{N} \in \mathbb{N}_\infty^*$ and $\{z_n\}_{n \in \mathrm{N}}$ a sequence of elements in $\mathbb{R}^m$, we denote by

$$\operatorname{acc} \{z_n\}_{n \in \mathrm{N}} \quad = \quad \{z \in \mathbb{R}^m | \exists \mathrm{N}' \subset \mathrm{N},\, \mathrm{N}' \in \mathbb{N}_\infty^*,\, z_n \to_{\mathrm{N}'} z\}$$

the accumulation points of $\{z_n\}_{n \in \mathrm{N}}$.

3 Economic model

By abuse of notation, we denote by $L = \{1, \ldots, L\}$, $I = \{1, \ldots, I\}$ and $J = \{1, \ldots, J\}$ the finite sets of types of consumption goods, consumers and firms, respectively. We assume that each type of agent $i \in I$ and $j \in J$ corresponds to a continuum of identical individuals indexed by compacts subsets $T_i \subset \mathbb{R}$ and $T_j \subseteq \mathbb{R}$, pairwise disjoint. The set of consumers and firms is denoted by

$$\mathcal{I} = \bigcup_{i \in I} T_i \quad \text{and} \quad \mathcal{J} = \bigcup_{j \in J} T_j,$$

respectively, and the type of producer $t \in \mathcal{J}$ is $j(t) \in J$, while the type of consumer $t \in \mathcal{I}$ is $i(t) \in I$.

In the following, each firm of type $j \in J$ is characterized by a production set $Y_j \subset \mathbb{R}^L$, and the aggregate production set for firms of type $j \in J$ is the convex hull of $\lambda(T_j)Y_j$, where $\lambda(\cdot)$ is the standard Lebesgue measure in $\mathbb{R}$. A production plan for a firm $t \in \mathcal{J}$ is denoted $y(t) \in Y_{j(t)}$, and the set of admissible production plans is

$$Y \quad = \quad \left\{ y \in L^1(\mathcal{J}, \cup_{j \in J} Y_j) \,|\, y(t) \in Y_{j(t)} \text{ a.e. } t \in \mathcal{J} \right\}.$$

Each consumer of type $i \in I$ is characterized by a consumption set $X_i \subset \mathbb{R}^L$, an endowment of resources $e_i \in \mathbb{R}^L$ and a strict preference correspondence $P_i : X_i \rightrightarrows X_i$. A consumption plan of an individual $t \in \mathcal{I}$ is denoted $x(t) \in X_{i(t)}$, and the set of admissible consumption plans is

$$X \quad = \quad \left\{ x \in L^1(\mathcal{I}, \cup_{i \in I} X_i) \,|\, x(t) \in X_{i(t)} \text{ a.e. } t \in \mathcal{I} \right\}.$$

The total initial resources of the economy is $e = \sum_{i \in I} \lambda(T_i) e_i \in \mathbb{R}^L$, and for $(i,j) \in I \times J$, $\theta_{ij} \in [0,1]$ is the share of type i consumer's in type j firms. As usual, we assume for every $j \in J$, $\sum_{i \in I} \lambda(T_i)\theta_{ij} = 1$. In addition, each consumer $t \in \mathcal{I}$ is initially endowed with an amount $m(t) \in \mathbb{R}_+$ of fiat money, with $m \in L^1(\mathcal{I}, \mathbb{R}_+)$.

An economy $\mathcal{E}$ is a collection

$$\mathcal{E} \quad = \quad \left((X_i, P_i, e_i)_{i \in I}, (Y_j)_{j \in J}, (\theta_{ij})_{(i,j) \in I \times J}, m, \{T_i\}_{i \in I}, \{T_j\}_{j \in J} \right), \tag{2}$$

and the feasible consumption-production plans of $\mathcal{E}$ are the elements of

$$A(\mathcal{E}) \quad = \quad \left\{ (x,y) \in X \times Y \,|\, \int_{\mathcal{I}} x(t)dt = \int_{\mathcal{J}} y(t)dt + e \right\}.$$

We will now define supply, demand and the equilibrium concepts. Let $p \in \mathbb{R}^L$, $q \in \mathbb{R}$ and K is a pointed cone[5] of $\mathbb{R}^L$, whose family is denoted $\mathcal{C}_L$. Using them,

$$\pi_j(p) = \lambda(T_j) \sup_{z \in Y_j} \; p \cdot z, \qquad S_j(p) = \operatorname*{arg\,max}_{z \in Y_j} \; p \cdot z,$$

and

$$\sigma_j(p, K) = \{ z \in S_j(p) \,|\, p \neq 0_L \Rightarrow (Y_j - \{z\}) \cap K = \{0_L\} \},$$

[5] We recall a cone $K \subseteq \mathbb{R}^L$ is said to be "pointed" when $-K \cap K = \{0_L\}$.

are, respectively, the profit, the Walras supply and the rationing supply of type $j \in J$ firms.[6]

The income of consumer $t \in \mathcal{I}$ is denoted by

$$w_t(p,q) = p \cdot e_{i(t)} + qm(t) + \sum_{j\in J} \theta_{i(t)j}\pi_j(p),$$

and the budget set is

$$B_t(p,q) = \left\{\xi \in X_{i(t)} |\, p\cdot \xi \leq w_t(p,q)\right\},$$

and for which we also denote by

$$d_t(p,q) = \left\{\xi \in B_t(p,q)|\, B_t(p,q)\cap P_{i(t)}(\xi) = \emptyset\right\}, \quad D_t(p,q) = \limsup_{(p',q')\to(p,q)} d_t(p',q'),$$

and

$$\delta_t(p,q,K) = \left\{\xi \in D_t(p,q)|\, \left(P_{i(t)}(\xi) - \{\xi\}\right) \subset K\right\}$$

the Walras, weak and rationing demand, respectively.[7]

Definition 3.1. *Given $(x,y,p,q) \in A(\mathcal{E}) \times \mathbb{R}^L \times \mathbb{R}_+$ and $K \in \mathcal{C}_L$, we call*

(a) (x,y,p,q) a Walras equilibrium with money of $\mathcal{E}$ if for a.e. $t \in \mathcal{I}$, $x(t) \in d_t(p,q)$ and for a.e. $t \in \mathcal{J}$, $y(t) \in S_{j(t)}(p)$,

(c) (x,y,p,q) a weak equilibrium of $\mathcal{E}$ if for a.e. $t \in \mathcal{I}$, $x(t) \in D_t(p,q)$ and for a.e. $t \in \mathcal{J}$, $y(t) \in S_{j(t)}(p)$,

(c) (x,y,p,q,K) a rationing equilibrium of $\mathcal{E}$ if for a.e. $t \in \mathcal{I}$, $x(t) \in \delta_t(p,q,K)$ and for a.e. $t \in \mathcal{J}$, $y(t) \in \sigma_t(p,K)$.

In order to define a sequence of discrete economies that approximates some economy $\mathcal{E}$, in the sequel we will use given sequences $\nu_h : \mathbb{N} \to \mathbb{N}$, $h = 1,\ldots,L$, such that $\lim_{n\to\infty} \nu_h(n) = \infty$, for all h. The family of subsets $\{M^n\}_{n\in\mathbb{N}}$ with

$$M^n \quad = \quad \left\{\xi = (\xi_1,\ldots,\xi_L) \in \mathbb{R}^L \,|\, (\nu_1(n)\xi_1,\ldots,\nu_L(n)\xi_L) \in \mathbb{Z}^L\right\},\ n \in \mathbb{N}, \tag{3}$$

then converges in the sense of Kuratowski-Painlevé to $\mathbb{R}^L$.

Definition 3.2. *We say that the sequence of economies $\{\mathcal{E}^n\}_{n\in\mathbb{N}}$, with*

$$\mathcal{E}^n \quad = \quad \left((X_i^n, P_i^n, e_i)_{i\in I}, (Y_j^n)_{j\in J}, (\theta_{ij})_{(i,j)\in I\times J}, m, \{T_i\}_{i\in I}, \{T_j\}_{j\in J}\right),\ n \in \mathbb{N}, \tag{4}$$

[6]The definition of the rationing supply we use here is less restrictive than in Florig and Rivera [8]. However, in the proofs of [8] it is only the requirement as imposed here which is used.

[7]Using definition in (1), for $t \in \mathcal{I}$, we have $D_t(p,q) = \bigcup_{\{(p_n,q_n)\to(p,q)\}} \limsup_{n\to\infty} d_t(p_n,q_n)$. As we will ensure that $d_t(\cdot)$ is closed valued and locally bounded, by Theorem 5.19 in Rockafellar and Wets [15], $D_t(\cdot)$ will be upper hemi-continuous while $d_t(\cdot)$ may fail to be upper hemi-continuous. See Florig and Rivera [9] for more details.

approximates the economy $\mathcal{E}$ if for all $n \in \mathbb{N}$, $i \in I$ and $j \in J$:

(i) $Y_j^n = Y_j \cap M^n \neq \emptyset$,

(ii) $X_i^n = X_i \cap M^n \neq \emptyset$,

(iii) P_i^n *is the restriction of* P_i *to* X_i^n.

Remark 3.1. The supply, demand and equilibrium concepts above defined for economy $\mathcal{E}$ can be readily adapted for economies $\mathcal{E}^n$, $n \in \mathbb{N}$. For that it is enough to replace Y_j, X_i and P_i by Y_j^n, X_i^n and P_i^n in the corresponding definitions.

The following assumptions will be used depending on the result to be established.

Assumption C. *For all* $(i, j) \in I \times J$, X_i *and* Y_j *are convex and compact polyhedral sets.*[8]

Assumption P. *For all* $i \in I$, P_i *is irreflexive and transitive, and has an open graph in* $X_i \times X_i$.

Assumption M. $m : \mathcal{I} \to \mathbb{R}_+$ *is bounded and for a.e.* $t \in \mathcal{I}$, $m(t) > 0$.

Assumption S. *For all* $i \in I$, $e_i \in \left(\mathrm{conv}X_i - \sum_{j\in J} \theta_{ij}\lambda(T_j)\,\mathrm{conv}Y_j\right)$.

Assumption SA. *For all* $i \in I$ $e_i \in \mathrm{int}\left(\mathrm{conv}X_i - \sum_{j\in J} \theta_{ij}\lambda(T_j)\,\mathrm{conv}Y_j\right)$.

Assumption A. *For all* $n \in \mathbb{N}$, $i \in I$ *and all* $j \in J$, $X_i = \mathrm{conv}X_i^n$ *and* $Y_j = \mathrm{conv}Y_j^n$.

Assumption F. *For all* $i \in I$ *and each face* F *of* X_i *such that*[9]

$$\left(\{e_i\} + \sum_{j\in J} \theta_{ij}\lambda(T_j)Y_j\right) \cap X_i \subseteq F,$$

the sequence $\{F \cap X_i^n\}_{n\in\mathbb{N}}$ *converges in the sense of Kuratowski-Painlevé to* F.

Assumption **F** requires that X_i^n restricted to the affine subspace for which the interiority assumption holds converges to X_i restricted to that affine subspace. This will be important to ensure that the budget set for a sequence of equilibria of the economies $\mathcal{E}^n$ converges to a budget set of the economy $\mathcal{E}$ for some limit of the price sequence considered.

The following proposition is an immediate consequence of Florig and Rivera [9]. For the proof it is enough to check that Assumption **C** on the economy $\mathcal{E}$ implies that the consumption and production sets of any economy of sequence $\{\mathcal{E}^n\}_{n\in\mathbb{N}}$ that approximates $\mathcal{E}$ are finite (i.e., the number of its elements is finite). The proposition ensures that the sequence of equilibria for which we study convergence do actually exist.

[8] That is, the convex hull of a finite number of vectors.

[9] For a convex compact polyhedron $P \subset \mathbb{R}^m$, a face is a set $F \subseteq P$ such that there exists $\psi \in \mathbb{R}^m$ with $F = \mathrm{argmax}\,\psi \cdot P$.

Proposition 3.1. *Suppose $\mathcal{E}$ satisfies Assumptions* **C**, **P**, **M** *and* **S**, *and let $\{\mathcal{E}^n\}_{n\in\mathbb{N}}$ be a sequence of economies approximating $\mathcal{E}$. For each $n \in \mathbb{N}$, there exists a rationing equilibrium $(x_n, y_n, p_n, q_n, K_n)$, with $q_n > 0$.*

4 Convergence under the Survival Assumption

In the next proposition, the survival assumption **SA** plays an important role in establishing the convergence to a Walras equilibrium. While this hypothesis is widely used, it is unrealistic, because it states that every consumer is initially endowed with a strictly positive quantity of every existing commodity. Typically, most consumers have a single commodity to sell (usually, their labor). In fact, it implies that all agents have the same level of income at equilibrium in the sense that they have all access to the same commodities.

Proposition 4.1. *Suppose $\mathcal{E}$ satisfies Assumptions* **C**, **P**, **M** *and* **SA**, *and let $\{\mathcal{E}^n\}_{n\in\mathbb{N}}$ be a sequence of economies approximating $\mathcal{E}$ satisfying Assumption* **A**. *For each $n \in \mathbb{N}$, let (x_n, y_n, p_n, q_n) be a weak equilibrium of $\mathcal{E}^n$, with $q_n > 0$ and $\| (p_n, q_n) \| = 1$. Then, there exists $\mathrm{N} \in \mathbb{N}^*_\infty$, such that the following hold:*

(a) $(p_n, q_n) \to_{\mathrm{N}} (p^*, q^*)$,

(b) there is $(x^*, y^*) \in A(\mathcal{E})$, such that for a.e. $t \in \mathcal{I}$, $x^*(t) \in \mathrm{acc}\{x_n(t)\}_{n\in\mathrm{N}}$, and for a.e. $t' \in \mathcal{J}$, $y^*(t') \in \mathrm{acc}\{y_n(t')\}_{n\in\mathrm{N}}$, with (x^*, y^*, p^*, q^*) a Walras equilibrium with fiat money for $\mathcal{E}$.

Moreover, if for a.e. $t \in \mathcal{I}$, $x^*(t) \in \mathrm{cl}\, P_{i(t)}(x^*(t))$, then (x^*, y^*, p^*) is a Walras equilibrium for $\mathcal{E}$

Proof. First note that $\{\mathcal{E}^n\}_{n\in\mathbb{N}}$ approximating $\mathcal{E}$ implies that for all $i \in I$, X_i^n converges to X_i. By Assumption **SA**, the smallest face of X_i containing

$$\left(\{e_i\} + \sum_{j\in J} \theta_{ij}\lambda(T_j)Y_j\right) \cap X_i$$

is X_i, which implies that Assumption **F** is satisfied. Therefore, all the assumptions of Theorem 5.1 below are satisfied. Assumption **SA** implies that for a hierarchic equilibrium $(x, y, \mathcal{P}, \mathcal{Q})$ with $\mathcal{P} = [\mathrm{p}_1, \ldots, \mathrm{p}_k]^{\mathrm{t}} \in \mathbb{R}^{k\times L}$ and $\mathcal{Q} = (\mathrm{q}_1, \ldots, \mathrm{q}_k)^{\mathrm{t}} \in \mathbb{R}^k_+$ (see definition in next section), such that $(x^*, y^*, \mathrm{p}_1, \mathrm{q}_1)$ is a Walras equilibrium with fiat money (cf Florig [6]). Moreover, if for a.e. $t \in \mathcal{I}$, $x^*(t) \in \mathrm{cl}\, P_{i(t)}(x^*(t))$, then standard arguments imply that $\mathrm{q}_1 = 0$. □

5 The general case

We now replace assumption **SA** by a more realistic one, i.e. we will assume that every consumer could decide not to exchange anything. We will not assume however that he could survive for very long without exchanging anything. In such a case the limit of a sequence of rationing equilibria will not necessarily be a Walras equilibrium, it will be a hierarchic equilibrium, which is a competitive equilibrium with a segmentation of individuals according to their level of wealth. When this segmentation consists of just one group, the hierarchic equilibrium reduces to a Walras equilibrium.

In the following, vectors of $\mathbb{R}^L$ are supposed to be columns, and for $k \in \mathbb{N}$, the matrix whose columns are $p_1, \ldots, p_k \in \mathbb{R}^L$ is denoted $[p_1, \ldots, p_k] \in \mathbb{R}^{L \times k}$. Given that, a hierarchic price for consumption goods is $\mathcal{P} = [p_1, \ldots, p_k]^{\mathrm{t}} \in \mathbb{R}^{k \times L}$, and the hierarchic value of $\xi \in \mathbb{R}^L$ is

$$\mathcal{P}\xi = (p_1 \cdot \xi, \ldots, p_k \cdot \xi)^{\mathrm{t}} \in \mathbb{R}^k.$$

Denoting by $\sup_{lex}$ the supremum with respect to $\leq_{lex}$, the lexicographic order[10] on $\mathbb{R}^L$, the hierarchic supply and the hierarchic profit of a firm of type $j \in J$ at $\mathcal{P}$ are

$$S_j(\mathcal{P}) = \{z \in Y_j \,|\, \forall z' \in Y_j,\ \mathcal{P}z' \leq_{lex} \mathcal{P}z\} \qquad \text{and} \qquad \pi_j(\mathcal{P}) = \lambda(T_j) \sup\nolimits_{lex}\{\mathcal{P}z \,|\, z \in Y_j\},$$

respectively, and given $\mathcal{Q} \in \mathbb{R}^k_+$, the hierarchic budget set of consumer $t \in \mathcal{I}$ is the set

$$B_t(\mathcal{P}, \mathcal{Q}) \quad = \quad \mathrm{cl}\left\{\xi \in X_{i(t)} \,|\, \mathcal{P}\xi \leq_{lex} \mathcal{P}e_{i(t)} + m(t)\,\mathcal{Q} + \sum_{j \in J} \theta_{i(t)j}\pi_j(\mathcal{P})\right\}.$$

Definition 5.1. *A collection $(x, y, \mathcal{P}, \mathcal{Q}) \in A(\mathcal{E}) \times \mathbb{R}^{k \times L} \times \mathbb{R}^k_+$ is a "hierarchic equilibrium" of the economy $\mathcal{E}$ if:*

(a) for a.e. $t \in \mathcal{J}$, $y(t) \in S_{j(t)}(\mathcal{P})$,

(b) for a.e. $t \in \mathcal{I}$, $x(t) \in B_t(\mathcal{P}, \mathcal{Q})$ and $P_{i(t)}(x(t)) \cap B_t(\mathcal{P}, \mathcal{Q}) = \emptyset$.

The number k in above expressions will be determined at the equilibrium. When $k = 1$ then the hierarchic equilibrium becomes a Walras equilibrium with money. The next theorem, a generalization of Proposition 4.1, is the main result of this paper. The proof is given in the Appendix.

Theorem 5.1. *Suppose $\mathcal{E}$ satisfies Assumptions* **C**, **P**, **M** *and* **S**, *and let $\{\mathcal{E}^n\}_{n \in \mathbb{N}}$ be a sequence of economies that approximates $\mathcal{E}$ and satisfying Assumptions* **A** *and* **F**. *For each $n \in \mathbb{N}$, let (x_n, y_n, p_n, q_n) be a weak equilibrium of $\mathcal{E}^n$, with $q_n > 0$ and $\| (p_n, q_n) \| = 1$. Then, there exists a hierarchic equilibrium $(x^*, y^*, \mathcal{P}, \mathcal{Q})$ for economy $\mathcal{E}$, with $\mathcal{P} = [\mathrm{p}_1, \ldots, \mathrm{p}_k]^{\mathrm{t}}$, $\mathcal{Q} = (\mathrm{q}_1, \ldots, \mathrm{q}_k)^{\mathrm{t}}$, $k \in \{1, \ldots, L\}$, such that for some $\mathrm{N} \in \mathbb{N}^*_\infty$ the following hold:*

(i) for each $n \in \mathrm{N}$, $p_n = \sum_{r=1}^k \varepsilon_r(n)\,\mathrm{p}_r$, with $\varepsilon_{r+1}(n)/\varepsilon_r(n) \to_{\mathrm{N}} 0$,

(ii) for a.e. $t \in \mathcal{I}$, $x^*(t) \in \mathrm{acc}\{x_n(t)\}_{n \in \mathrm{N}}$, and for a.e. $t' \in \mathcal{J}$, $y^*(t') \in \mathrm{acc}\{y_n(t')\}_{n \in \mathrm{N}}$.

[10] For $(s, t) \in \mathbb{R}^m \times \mathbb{R}^m$, we recall $s \leq_{lex} t$, if $s_r > t_r$, $r \in \{1, \ldots, m\}$ implies that $\exists \rho \in \{1, \ldots, r-1\}$ such that $s_\rho < t_\rho$. We write $s <_{lex} t$ if $s \leq_{lex} t$, but not $t \leq_{lex} s$.

Remark 5.1. As a rationing equilibrium is a weak equilibrium (see Definition 3.1), Theorem 5.1 remains valid when using a sequence of rationing equilibria instead of a sequence of weak equilibria as stated.

6 Appendix

For the proof of Theorem 5.1 we need some additional definitions and technical lemmata. The following lemma is proven in Florig and Rivera [9].

Lemma 6.1. *For every sequence $\psi : \mathbb{N} \to \mathbb{R}^m \setminus \{0_m\}$ there are an integer $k \in \{1, \dots, m\}$, $\mathrm{N} \in \mathbb{N}^*_\infty$, a set of two-by-two orthonormal vectors $\{\psi_1, \dots, \psi_k\} \subseteq \mathbb{R}^m$ and sequences $\varepsilon_r : \mathrm{N} \to \mathbb{R}_{++}$, $r \in \{1, \dots, k\}$, such that the following hold:*

(a) for all $r \in \{1, \dots, k-1\}$, $\varepsilon_{r+1}(n)/\varepsilon_r(n) \to_{\mathrm{N}} 0$,

(b) for all $n \in \mathrm{N}$, $\psi(n) = \sum_{r=1}^{k} \varepsilon_r(n)\psi_r$.

In the following, we say the collection $\{\{\psi_r, \varepsilon_r\}_{r=1,\dots,k}, \mathrm{N}\}$ is a lexicographic decomposition of $\{\psi(n)\}_{n\in\mathrm{N}}$, and for $r \in \{0, \dots, k\}$ we denote

$$\Psi(r) \quad = \quad \begin{cases} [\psi_1, \dots, \psi_r]^{\mathrm{t}} & \text{if} \quad r > 0, \\ 0_m^{\mathrm{t}} & \text{if} \quad r = 0. \end{cases} \tag{5}$$

Furthermore, for $z \in \mathbb{R}^m$ and $r > 0$, we also denote $\Psi(r)z = (\psi_1 \cdot z, \dots, \psi_k \cdot z)^{\mathrm{t}} \in \mathbb{R}^r$, and for $Z \subseteq \mathbb{R}^m$ we set $\Psi(r)Z = \{\Psi(r)z \,|\, z \in Z\}$.

The following lemmata refer to a sequence and lexicographic decomposition as in above lemma. Parts (i) and (ii) of Lemma 6.2 were proven in Florig and Rivera [9], while part (iii) is a direct corollary of part (ii) coupled with the observation that for any $\psi \in \mathbb{R}^m$ and finite set of points $Z \subset \mathbb{R}^m$, $\operatorname{conv} \operatorname{argmax} \psi \cdot Z = \operatorname{argmax} \psi \cdot \operatorname{conv} Z$.

Lemma 6.2.

(i) For all $z \in \mathbb{R}^m$ there exists $\bar{n} \in \mathbb{N}$ such that for all $n > \bar{n}$ with $n \in \mathrm{N}$:

$$\Psi(k)\, z \leq_{lex} 0_k \quad \Longleftrightarrow \quad \psi(n) \cdot z \leq 0.$$

(ii) If $Z \subseteq \mathbb{R}^m$ is a finite set, then there exists $\bar{n} \in \mathbb{N}$ such that for all $n > \bar{n}$ with $n \in \mathrm{N}$:

$$\operatorname{argmax}_{lex} \Psi(k)\, Z \quad = \quad \operatorname{argmax}\, \psi(n) \cdot Z.$$

(iii) If $Z \subseteq \mathbb{R}^m$ is a convex and compact polyhedron, then there exists $\bar{n} \in \mathbb{N}$ such that for all $n > \bar{n}$ with $n \in \mathrm{N}$:

$$\operatorname{argmax}_{lex} \Psi(k)\, Z \quad = \quad \operatorname{argmax}\, \psi(n) \cdot Z.$$

Both parts (ii) and (iii) in Lemma 6.2 remain valid when replacing $\operatorname{argmax}_{lex}$ by $\operatorname{argmin}_{lex}$.

Lemma 6.3. *Let $Z \subset \mathbb{R}^m$ be a convex and compact polyhedron, for which we define*

$$\rho = \max\left\{r \in \{0,\dots,k\} \,|\, \min_{lex} \Psi(r)Z = 0_{\max\{1,r\}}\right\} \quad \text{and} \quad \mathcal{F} = \operatorname{argmin}_{lex} \Psi(\rho)Z.$$

The following hold true.

(i) $\limsup_{n\to\infty}\{z \in Z \,|\, \psi(n)\cdot z \le 0\} \subseteq \operatorname{cl}\{z \in Z \,|\, \Psi(k)z \le_{lex} 0_k\}$.

(ii) Suppose that $\min_{lex} \Psi(k)Z <_{lex} 0_k$*, and let* $\{Z_n\}_{n\in\mathrm{N}} \subset \mathbb{R}^m$ *such that*

$$\lim_{n\to\infty} Z_n = Z \quad \text{and} \quad \lim_{n\to\infty} (Z_n \cap \mathcal{F}) = Z \cap \mathcal{F}.$$

Then

$$\operatorname{cl}\{z \in Z \,|\, \Psi(k)z \le_{lex} 0_k\} \quad \subseteq \quad \liminf_{n\to\infty}\{z \in Z_n \cap \mathcal{F} \,|\, \psi(n)\cdot z < 0\}.$$

Proof. In the following, for any $\mathrm{N} \in \mathbb{N}^*_\infty$ and $n' \in \mathbb{N}$, we will use the notation $\mathrm{N}_{n'} = \{n \in \mathrm{N} \,|\, n > n'\}$.

Part (i). Let $\bar z \in \operatorname{argmin}_{lex} \Psi(k)Z$ and assume that $\limsup_{n\to\infty}\{z \in Z|\psi(n)\cdot z \le 0\} \ne \emptyset$, since otherwise the result is trivial. Hence, for z^* in that subset, there is $\overline{\mathrm{N}} \in \mathbb{N}^*_\infty$ and $\{z_n\}_{n\in\overline{\mathrm{N}}} \subset Z$ such that $z_n \to_{\overline{\mathrm{N}}} z^*$ and for all $n \in \overline{\mathrm{N}}$, $\psi(n)\cdot z_n \le 0$. By Lemma 6.2, part (iii), there exists $n_1 \in \mathbb{N}$ such that for all $n \in \overline{\mathrm{N}}_{n_1}$, we have

$$\operatorname{argmin}_{lex} \Psi(k)\, Z = \operatorname{argmin} \psi(n)\cdot Z.$$

As for all $n \in \overline{\mathrm{N}}_{n_1}$,

$$\psi(n)\cdot \bar z = \min \psi(n)\cdot Z \le \psi(n)\cdot z_n \le 0,$$

we have by part (i) of Lemma 6.2 that $\Psi(k)\,\bar z \le_{lex} 0_k$.

Let $\sigma = \max\left\{r \in \{0,\dots,k\} \,|\, \Psi(r)z^* = 0_{\max\{1,r\}}\right\}$. If $\Psi(k)z^* \le_{lex} 0_k$, then the conclusion is trivial. Therefore, we assume $\Psi(k)z^* >_{lex} 0_k$, which implies that $\sigma < k$ and $\psi_{\sigma+1}\cdot z^* = \delta > 0$.

Case 1. $\rho < \sigma$.

As $\rho < \sigma$, we have $\rho < k$, $\Psi(\rho+1)\,\bar z <_{lex} 0_{\rho+1}$ and $\Psi(\rho+1)\,z^* = 0_{\rho+1}$. Therefore, for all $\mu \in [0,1[$,

$$\Psi(\rho+1)\,(\mu\bar z + (1-\mu)z^*) <_{lex} 0_{\rho+1}.$$

Hence $\Psi(k)\,(\mu\bar z + (1-\mu)z^*) <_{lex} 0_k$, implying that $z^* \in \operatorname{cl}\{z \in Z \,|\, \Psi(k)z \le_{lex} 0_k\}$.

Case 2. $\rho \ge \sigma$.

As $\rho \ge \sigma$, for all $r \in \{1,\dots,\sigma\}$, $\psi_r\cdot\bar z = \psi_r\cdot z^* = 0$. Then $\{\bar z, z^*\} \subseteq \operatorname{argmin}_{lex} \Psi(\sigma)Z$. For $n \in \overline{\mathrm{N}}$ we set

$$\psi^*(n) = \sum_{r=1}^{\sigma} \varepsilon_r(n)\psi_r,$$

with $\psi^*(n) = 0$ when $\sigma = 0$. By part (ii) in Lemma 6.2 there exists $n_2 > n_1$ such that for all $n \in \overline{\mathrm{N}}_{n_2}$,

$$0 = \psi^*(n) \cdot \bar{z} = \psi^*(n) \cdot z^* \leq \psi^*(n) \cdot z_n.$$

For $n \in \overline{\mathrm{N}}$, we set

$$a_n = \sum_{r=1}^{\sigma+1} \varepsilon_r(n)\, \psi_r \cdot z_n \quad \text{and} \quad b_n = \frac{\varepsilon_{\sigma+2}(n)}{\varepsilon_{\sigma+1}(n)} \sum_{r=\sigma+2}^{k} \frac{\varepsilon_r(n)}{\varepsilon_{\sigma+2}(n)}\, \psi_r \cdot z_n,$$

with $b_n = 0$ if $\sigma + 1 = k$. By the fact that $\{z_n\}_{n\in\overline{\mathrm{N}}}$ remains in a compact set, there exists $n_3 > n_2$ such that for all $n \in \overline{\mathrm{N}}_{n_3}$, on the one hand

$$a_n \geq \varepsilon_{\sigma+1}(n)\, \psi_{\sigma+1} \cdot z_n > \varepsilon_{\sigma+1}(n) \frac{\delta}{2},$$

and, on the other hand, since b_n converges to zero,

$$b_n \in \frac{1}{4}[-\delta, \delta].$$

Therefore, for all $n \in \overline{\mathrm{N}}_{n_3}$,

$$0 \geq \psi(n) \cdot z_n = a_n + \varepsilon_{\sigma+1}(n) b_n \geq \varepsilon_{\sigma+1}(n) \frac{\delta}{4},$$

contradicting $\delta > 0$, hence concluding the proof of part (i).

Part (ii). Let $\bar{z} \in \operatorname{argmin}_{lex} \Psi(k)Z$. By the fact that $\min_{lex} \Psi(k)Z <_{lex} 0_k$, we have $\rho < k$ and $\psi_{\rho+1} \cdot \bar{z} < 0$. Let $\zeta \in \operatorname{cl}\{z \in Z \,|\, \Psi(k)z \leq_{lex} 0_k\}$. Then, for $\varepsilon \in]0, 1]$ there exists $\zeta_\varepsilon \in \mathbb{B}(\zeta, \varepsilon/2) \cap Z$ such that $\Psi(k)\zeta_\varepsilon \leq_{lex} 0_k$. By the convexity of Z, for $\mu \in]0, \varepsilon/2[$ it follows that

$$z_\varepsilon = (1 - \mu)\zeta_\varepsilon + \mu\bar{z} \in Z \cap \mathbb{B}(\zeta, \varepsilon),$$

and then

$$\Psi(k)\bar{z} \leq_{lex} \Psi(k)z_\varepsilon \leq_{lex} \Psi(k)\zeta_\varepsilon \leq_{lex} 0_k.$$

The definition of ρ implies $\Psi(\rho)\bar{z} = 0_{\max\{1,\rho\}}$ and therefore we have also

$$\Psi(\rho)z_\varepsilon = \Psi(\rho)\zeta_\varepsilon = 0_{\max\{1,\rho\}}.$$

This coupled with the fact that $\rho < k$ implies

$$\psi_{\rho+1} \cdot \bar{z} \leq \psi_{\rho+1} \cdot z_\varepsilon \leq \psi_{\rho+1} \cdot \zeta_\varepsilon \leq 0$$

and $\psi_{\rho+1} \cdot \bar{z} < 0$. Since $\psi_{\rho+1} \cdot \zeta_\varepsilon \leq 0$, we also have $\delta = \psi_{\rho+1} \cdot z_\varepsilon < 0$. Therefore $\Psi(\rho+1)z_\varepsilon <_{lex} 0_{\rho+1}$. Now, we have established that $\Psi(k)z_\varepsilon <_{lex} 0$, $z_\varepsilon \in \mathcal{F}$ and $z_\varepsilon \in \mathbb{B}(\zeta, \varepsilon)$.

Let us now consider $\{z_n\}_{n\in\mathbb{N}} \subseteq \mathcal{F} \cap Z_n$ with $z_n \to_{\mathbb{N}} z_\varepsilon$. We observe that

$$\psi(n) \cdot z_n = \sum_{r=1}^{k} \varepsilon_r(n)\psi_r \cdot z_n = \varepsilon_{\rho+1}(n)\,(\alpha_n + \beta_n),$$

where

$$\alpha_n = \frac{1}{\varepsilon_{\rho+1}(n)} \sum_{r=1}^{\rho+1} \varepsilon_r(n)\psi_r \cdot z_n \quad \text{and} \quad \beta_n = \frac{1}{\varepsilon_{\rho+1}} \sum_{r=\rho+2}^{k} \varepsilon_r(n)\psi_r \cdot z_n,$$

with $\beta_n = 0$ if $\rho + 1 = k$. Given that, for all $n \in \mathbb{N}$, $\Psi(\rho)z_n = 0_{\max\{1,\rho\}}$, and as β_n converges to 0 and $\delta < 0$, there exists $\bar{n}$ such that for all $n \in \mathbb{N}$ with $n > \bar{n}$, $\alpha_n < \delta/2$ and $\beta_n < -\delta/4$ and therefore $\alpha_n + \beta_n < \delta/4 < 0$. All of this implies that for all $n \in \mathbb{N}$ with $n > \bar{n}$,

$$\psi(n) \cdot z_n = \varepsilon_{\rho+1}(n)\,(\alpha_n + \beta_n) < \varepsilon_{\rho+1}(n)\,\delta/4 < 0.$$

Therefore, for $\zeta \in \text{cl}\{z \in Z \mid \Psi(k)z \leq_{lex} 0_k\}$ and $\varepsilon \in]0,1]$, we have that

$$z_\varepsilon \in \mathbb{B}(\zeta, \varepsilon) \cap \liminf_{n\to\infty}\{z \in Z_n \cap \mathcal{F} \mid \psi(n) \cdot z < 0\},$$

and then, since the lim inf above is a closed set,[11] $\zeta \in \liminf_{n\to\infty}\{z \in Z_n \cap \mathcal{F} \mid \psi(n) \cdot z < 0\}$. $\square$

6.1 Proof of Theorem 5.1

In the following, we use a sequence $(x_n, y_n, p_n, q_n)_{n\in\mathbb{N}}$ of weak equilibria with $q_n > 0$ of the economy $\mathcal{E}^n$ (which exists by Proposition 3.1, considering that a rationing equilibrium is a weak equilibrium). For economy $\mathcal{E}^n$, the supply correspondence of type $j \in J$ firms is denoted $S_j^n(\cdot)$, while $B_t^n(\cdot)$ and $D_t^n(\cdot)$ denote the budget and weak demand correspondences for consumer $t \in \mathcal{I}$.[12] We can assume without loss of generality that for all $t \in \mathcal{I}$, $x_n(t) \in D_t^n(p_n, q_n)$ and all $t \in \mathcal{J}$, $y_n(t) \in S_j^n(p_n)$.[13]

The following proposition has been proven in Florig and Rivera [9].

Proposition 6.1. *Let $n \in \mathbb{N}$, $(p,q) \in \mathbb{R}^L \times \mathbb{R}_{++}$ and assume $m(t) > 0$, then*

$$D_t^n(p,q) \quad = \quad \left\{\xi \in B_t^n(p,q) \mid \inf\left\{p \cdot P_{i(t)}^n(\xi)\right\} \geq w_t^n(p,q),\ \xi \notin \text{conv}\, P_{i(t)}^n(\xi)\right\}.$$

In the remaining of this paper we split the proof of theorem into six steps.

[11] See, for example, Proposition 4.4 in Rockafellar and Wets [15].

[12] That is, for $n \in \mathbb{N}$, $S_j^n(p) = \arg\max_{z\in Y_j^n}\ p \cdot z$, and denoting $w_t^n(p,q) = p \cdot e_{i(t)} + qm(t) + \sum_{j\in J} \theta_{i(t)j} \max p \cdot Y_j^n$ and $B_t^n(p,q) = \left\{\xi \in X_{i(t)}^n | \, p \cdot \xi \leq w_t^n(p,q)\right\}$, then $d_t^n(p,q) = \left\{\xi \in B_t^n(p,q) | \, B_t^n(p,q) \cap P_{i(t)}^n(\xi) = \emptyset\right\}$ and therefore $D_t^n(p,q) = \limsup_{(p',q')\to(p,q)} d_t^n(p',q')$.

[13] Since a countable union of negligible sets is negligible, we could restrict the sequel to an appropriate subset of full Lebesgue measure. Here, as the consumption and production sets are finite for each $n \in \mathbb{N}$, we could also adjust the sequence (x_n, y_n) such that for all $t \in \mathcal{I}$, $x_n(t) \in D_t^n(p_n, q_n)$ and all $t \in \mathcal{J}$, $y_n(t) \in S_j^n(p_n)$ while maintaining $(x_n, y_n) \in A(\mathcal{E}^n)$.

Step 1 . *Hierarchic price.*

Since $\|(p_n, q_n)\| = 1$, $n \in \mathbb{N}$, from Lemma 6.1 there exist

$$\{\{(\mathrm{p}_r, \mathrm{q}_r), \varepsilon_r\}_{r=1,\dots,k}, \mathrm{N}\}$$

a lexicographic decomposition of the sequence $\{\psi(n) = (p_n, q_n)\}_{n\in\mathrm{N}}$. In the sequel, without loss of generality, we identify that subset N with $\mathbb{N}$, and we denote

$$\mathcal{P} = [\mathrm{p}_1, \dots, \mathrm{p}_k]^{\mathrm{t}} \quad \text{and} \quad \mathcal{Q} = (\mathrm{q}_1, \dots, \mathrm{q}_k)^{\mathrm{t}},$$

and for $r \in \{1, \dots, k\}$, we set $\mathcal{P}(r) = [\mathrm{p}_1, \dots, \mathrm{p}_r]^{\mathrm{t}}$ and $\mathcal{Q}(r) = (\mathrm{q}_1, \dots, \mathrm{q}_r)^{\mathrm{t}}$.

Step 2. *Supply: For all $t \in \mathcal{J}$, $\limsup_{n\to\infty} S^n_{j(t)}(p_n) \subseteq S_{j(t)}(\mathcal{P})$.*

As for all $j \in J$, by Lemma 6.2 there exists $n_j \in \mathbb{N}$ such that for all $n > n_j$,

$$S_j(p_n) = S_j(\mathcal{P}) = \mathrm{argmax}_{lex}\mathcal{P}Y_j.$$

For all $n \in \mathbb{N}$ and all $j \in J$, $\mathrm{conv}Y^n_j = Y_j$, $S^n_j(p_n) \subseteq S_j(p_n) = \mathrm{conv}S^n_j(p_n)$. This implies that for all $n > n_J = \max\{n_j,\, j = 1, \dots, J\}$, and all $t \in \mathcal{J}$,

$$S^n_{j(t)}(p_n) \subseteq S_{j(t)}(\mathcal{P}) = \mathrm{conv}S^n_{j(t)}(p_n),$$

hence concluding the proof of this Step.

Step 3. *Income.*

For the sequel, for all $j \in J$, let $\zeta_j \in \mathrm{argmax}_{lex}\mathcal{P}Y_j$, and for all $i \in I$, we set

$$z_i = e_i + \sum_{j\in J} \theta_{ij}\lambda(T_j)\zeta_j.$$

By Step 2, for all $t \in \mathcal{I}$ and all $n > n_J$, $w_t(p_n, q_n) = p_n \cdot z_{i(t)} + q_n m(t)$.

Step 4. *Budget: For all $t \in \mathcal{I}$,*

$$\limsup_{n\to\infty} B_t(p_n, q_n) \quad \subseteq \quad B_t(\mathcal{P}, \mathcal{Q}).$$

Furthermore, if $m(t) > 0$ then

$$B_t(\mathcal{P}, \mathcal{Q}) \quad \subseteq \quad \liminf_{n\to\infty} \left\{ x \in X^n_{i(t)} \,|\, p_n \cdot x < w_t(p_n, q_n) \right\}.$$

Using z_i from Step 3, the first inclusion is a straightforward consequence of part (i) of Lemma 6.3 applied to $Z = \big(X_{i(t)} - z_i\big) \times \{-m(t)\}$. Indeed, note that for all $n \in \mathbb{N}$, $n > n_J$, and all $x'_n(t) \in B_t(p_n, q_n)$ we have $\psi_n \cdot z_n \leq 0$ with $z_n = (x'_n(t) - z_{i(t)}, -m(t))$ and $\psi(n) = (p_n, q_n)$.

For the second inclusion, for $t \in \mathcal{I}$ and $n \in \mathbb{N}$, we set

$$\rho = \max\{r \in \{0,\dots,k\} \,|\, \min_{lex}\mathcal{P}(r)X_{i(t)} = 0_{\max\{1,r\}}\} \quad \text{and} \quad \mathcal{F} = \min_{lex}\mathcal{P}(\rho)X_{i(t)}.$$

Assumption **S** coupled with the observation $m(t)\mathcal{Q} >_{lex} 0_k$ implies

$$\min_{lex} \mathcal{P}(\big(X_{i(t)} - z_{i(t)}\big) - m(t)\mathcal{Q} <_{lex} 0_k \quad \text{and} \quad m(t)\mathcal{Q}(\rho) = 0_{\max\{1,\rho\}}.$$

Therefore, producers profit maximization and assumption **S** implies

$$\left(\{e_{i(t)}\} + \sum_{j\in J} \theta_{i(t)j}\lambda(T_j)\, Y_j\right) \cap X_{i(t)} \subseteq \mathcal{F}.$$

By part (iii) of Lemma 6.2 we observe that $\mathcal{F}$ is a face of $X_{i(t)}$, and then, by Assumption **F** it follows that

$$\lim_{n\to\infty} X^n_{i(t)} \cap \mathcal{F} = X_{i(t)} \cap \mathcal{F}.$$

By part (ii) of Lemma 6.3

$$B_t(\mathcal{P},\mathcal{Q}) \quad \subseteq \quad \liminf_{n\to\infty} \left\{x \in X^n_{i(t)} \cap \mathcal{F} \,|\, p_n \cdot (x - z_{i(t)}) < q_n m(t)\right\},$$

and since

$$\liminf_{n\to\infty} \left\{x \in X^n_{i(t)} \cap \mathcal{F} \,|\, p_n \cdot (x - z_{i(t)}) < q_n m(t)\right\} \quad \subseteq \quad \liminf_{n\to\infty} \left\{x \in X^n_{i(t)} \,|\, p_n \cdot x < w_t(p_n,q_n)\right\}$$

the second inclusion holds true.

Step 5. *Demand: For all $t \in \mathcal{I}$ with $m(t) > 0$ and all $x^*(t) \in \operatorname{acc}\{x_n(t)\}_{n\in\mathbb{N}}$ we have*

$$P_{i(t)}(x^*(t)) \cap B_t(\mathcal{P},\mathcal{Q}) = \emptyset.$$

Let $t \in \mathcal{I}$ such that $m(t) > 0$ and choose $\mathrm{N}(t) \in \mathbb{N}^*_\infty$ such that $x_n(t) \to_{\mathrm{N}(t)} x^*(t)$ and for all $n \in \mathrm{N}(t)$, $n > n_J$. By contraposition, assume that there is $\xi \in P_{i(t)}(x^*(t)) \cap B_t(\mathcal{P},\mathcal{Q})$.

By Step 5, there exists $\bar{n}_1 > n_J$ and $\xi_n \to_{\mathbb{N}} \xi$ such that for all $n > \bar{n}_1$ with $n \in \mathbb{N}$,

$$p_n \cdot (\xi_n - z_{i(t)}) - q_n m(t) < 0 \quad \text{and} \quad \xi_n \in X^n_{i(t)}.$$

As the graph of $P_{i(t)}$ is open, there exists $\bar{n}_2 > \bar{n}_1$ such that for all $n > \bar{n}_2$ with $n \in \mathbb{N}$,

$$p_n \cdot (\xi_n - z_{i(t)}) - q_n m(t) < 0 \quad \text{and} \quad \xi_n \in X^n_{i(t)} \cap P_{i(t)}(x^*(t)),$$

and again as the graph of $P_{i(t)}$ is open, we can choose $\bar{n}_3 > \bar{n}_2$ such that for all $n > \bar{n}_3$ with

$n \in N(t)$,

$$p_n \cdot (\xi_n - z_{i(t)}) - q_n m(t) < 0 \quad \text{and} \quad \xi_n \in P^n_{i(t)}(x_n(t)).$$

As $q_n m(t) > 0$, the last fact contradicts $x_n(t) \in D^n_t(p_n, q_n)$ for all $n > \bar{n}_3$ with $n \in N(t)$ (see Proposition 6.1).

Step 6. *Equilibrium allocation.*

Using Fatou's lemma in Artstein [2], there exists $(x^*, y^*) \in A(\mathcal{E})$ such that for a.e. $t \in \mathcal{I}$ and a.e. $t' \in \mathcal{J}$, $x^*(t) \in \text{acc}\{x_n(t)\}_{n\in\mathbb{N}}$ and $y^*(t') \in \text{acc}\{y_n(t')\}_{n\in\mathbb{N}}$. By Step 2, for a.e. $t \in \mathcal{J}$, $y^*(t) \in S_{j(t)}(\mathcal{P})$, and by Steps 4 and 5, for a.e. $t \in \mathcal{I}$, $x^*(t) \in B_t(\mathcal{P}, \mathcal{Q})$ and $P_{i(t)}(x^*(t)) \cap B_t(\mathcal{P}, \mathcal{Q}) = \emptyset$.

References

[1] Arrow, K.J. and G. Debreu (1954): "Existence of an Equilibrium for a Competitive Market," *Econometrica*, 22, 265-290.

[2] Artstein, A. (1979): "A Note on Fatou's Lemma in Several Dimensions," *Journal of Mathematical Economics*, 6 , 277-282.

[3] J.P. Aubin and H. Frankowska, "Set-valued analysis", Birkhäuser, Basel, 1990.

[4] Bobzin, H. (1998): *Indivisibilities: Microecomic Theory with Respect to Indivisible Goods and Factors*, Physica Verlag, Heidelberg.

[5] Danilov, V.I. and A.I. Sotskov, 1990, "A Generalized Economic Equilibrium", *Journal of Mathematical Economics*, 19, 341-356

[6] Florig, M. (2001): "Hierarchic Competitive Equilibria," *Journal of Mathematical Economics*, 35(4), 515-546.

[7] Florig, M. (2003): "Arbitrary Small Indivisibilities", *Economic Theory*, 22, 831-843.

[8] Florig, M. and J. Rivera (2010):"Core equivalence and welfare properties without divisible goods". *Journal of Mathematical Economics*, 46, 467-474.

[9] Florig, M. and J. Rivera (2015):"Existence of a competitive equilibrium when all goods are indivisible". Serie de documentos de trabajo, SDT403, Departamento de Economía, Universidad de Chile.

[10] Gay, A. 1978, "The Exchange Rate Approach to General Economic Equilibrium", Economie Appliquée, Archives de l' I.S.M.E.A. XXXI, nos. 1,2, 159-174

[11] Kajii, A. (1996): "How to Discard Non-Satiation and Free-Disposal with Paper Money," *Journal of Mathematical Economics*, 25, 75-84.

[12] Konovalov, A. (2005): "The core of an economy with satiation", *Economic Theory*, 25, 711-719.

[13] Marakulin, V. (1990): "Equilibrium with Nonstandard Prices in Exchange Economies," in *Optimal decisions in market and planned economies*, edited by R. Quandt and D. Triska, Westview Press, London, 268-282.

[14] Mertens, J.F. 2003, "The limit-price mechanism", *Journal of Mathematical Economics*, 39, 433-528

[15] Rockafellar, R. and Wets, R. (1998): *Variational Analysis*. Springer.