Indivisible goods and fiat money*

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Abstract

Although fiat money is useless in standard Arrow-Debreu models, in this paper we will show that this general conclusion does not hold true when goods are indivisible. In our setting, fiat money is valuable because it facilitates exchange, its price will always be positive and equilibrium allocations will change with the distribution of fiat money even though it does not directly yield utility through consumer preferences. Since a Walrasian equilibrium does not necessarily exist when goods are indivisible, a new equilibrium concept - called a rationing equilibrium - is introduced and its existence is proven under weak assumptions on the economy. A rationing equilibrium is a Walrasian equilibrium for all generic fiat money distributions.

Keywords: competitive equilibrium, indivisible goods, fiat money.

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1 Introduction

Most economic models assume that goods are perfectly divisible. The rational behind this assumption is that the commodities usually considered are *almost perfectly divisible* in the sense that the minimal unit of the good is insignificant enough so that its indivisibility can be neglected. In this way one should be able to approximate an economy, with a *small enough level* of indivisibility of goods, by some idealized economy where goods are perfectly divisible. The competitive equilibrium in this idealized economy should thus be an approximation of some competitive outcome of the economy with indivisible goods.

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The question that arises is what the Walrasian equilibrium with perfectly divisible goods is supposed to approximate - simply the Walrasian equilibrium of an economy with indivisible goods? It is well known that a Walrasian equilibrium may fail to exist in the absence of perfectly divisible goods, and that even the core may be empty (see Henry (1970) and Shapley and Scarf (1974) respectively). These facts are certainly due to some economic phenomena which is not taken care for with the standard approach. Consequently, a richer notion of competitive equilibrium is needed, which exists even when goods are indivisible. This new notion will be called *a rationing equilibrium*.

In order to define the rationing equilibrium concept, we will develop a model where (i) goods are indivisible at the individual level but perfectly divisible at the aggregate level of the economy; (ii) fiat money is used only to facilitate the exchange among consumers; and (iii) we introduce a new notion of demand which will be an upper semi-continuous correspondence in our framework.

Concerning (i), we consider a model where there is a finite number of types of consumers, and for each type, there is a continuum of individuals. So if an individual owns a house or not for example, it is not negligible to him but it is to the economy as a whole. Note also that if the initial endowment of an individual was not negligible at the aggregate level, it might be hard to justify that he acts as a price taker.

With respect to (ii), it is clear that in the presence of divisible goods, it could be difficult for agents to execute net-exchanges worth exactly zero[1]. Similarly to Drèze and Müller (1980) we introduce a slack parameter in the economy, which in our case can be identified as fiat money. Its only role will be to facilitate the exchange of goods among individuals. Indeed, fiat money has no intrinsic value whatsoever, since in our model it does not enter in consumers' preferences.

Finally, regarding point (iii), we point out that in the presence of indivisible goods, the Walrasian demand is, in general, not an upper semi-continuous correspondence and so, a Walrasian equilibrium do not necessarily exist. Technically speaking a different notion of demand may overcome this problem. In Section 2 below we give a heuristic argument why it is natural to consider a different notion of demand, which coincides in the convex case with the standard approach. This will be examined in detail in Section 4 where we analyze its interpretation and properties.

Fiat money will play a relevant role in an economy whenever its price is strictly positive, because in these cases it will modify the consumers' budgetary set and therefore permits equilibrium allocations that can otherwise not be reached (see example (i) in Section 2).

As we already know, the role of fiat money is a crucial problem in monetary economic theory. For example, Samuelson (1958), Balasko, Cass and Shell (1980), Balasko and Shell (1981), among others, developed an infinite horizon model with overlapping generations in order to demonstrate the essentiality of fiat money in an economy, whereas Bewley (1983), Gale and Hellwig (1984) consider infinitely lived agents. In a static or finite horizon model, one may consider lump-sum money taxation with zero

[1]Already Adam Smith (1976) saw in money the possibility to facilitate exchange of indivisible goods as one of its crucial roles. See also Kiyotaki and Wright (1989) for a more detailed model on this relevant aspect of fiat money in economy.

total money supply (Lerner (1947), Balasko and Shell (1986)). Clower (1967) proposed a cash in advance constraint to study similar problems (complementarily, see Dubey and Geanakoplos (1992), Drèze and Polemarchakis (2000)). Another approach takes advantage of the existence of frictions in economy that help us to ensure the essentiality of money in economy. To get details on these types of models see, for instance, Diamond (1984), Shi (1995), Kocherlakota (1998), and Rocheteau and Wright (2005) as a general reference.

In spite of all the above, the introduction of fiat money into the Arrow-Debreu model may be necessary in much simpler settings than the aforementioned. For example, if the non-satiation assumption does not hold, for any given price, consumers may demand a commodity bundle in the interior of their budget set. Therefore a Walrasian equilibrium may fail to exist. However, without the non-satiation assumption one may establish the existence of equilibria by allowing for the possibility that some agents spend more than the value of their initial endowment. This generalization of the Walrasian equilibrium is called *dividend equilibrium* or *equilibrium with slack* (see Makarov (1981), Balasko (1982), Aumann and Drèze (1986) and Mas-Colell (1992) among others). This concept was first introduced in a fixed price setting by Drèze and Müller (1980). Kajii (1996) shows that this dividend approach is equivalent to considering Walrasian equilibria with an additional commodity called fiat money. In that setting, fiat money can be consumed in positive quantities, but preferences are independent of its consumption. Thus, if local non-satiation holds, fiat money is worthless and we are back in the Arrow-Debreu setting. However, if the satiation problem occurs, fiat money must have a positive price in equilibrium. If a consumer does not want to spend his entire income on consumption goods, he can satisfy his budget constraint with equality by buying fiat money, if this fiat money has a positive price.

In our approach, all goods are indivisible and therefore local non-satiation never holds. This is the reason why fiat money always has a positive price and not only occasionally, as in the standard approach without non-satiation.

The main result of this paper is the existence of a rationing equilibrium with a strictly positive price for fiat money (Theorem 5.1). Moreover, as a consequence of this result, we will prove that if each consumer is initially endowed with a different amount of fiat money (strictly positive), then a Walrasian equilibrium with a strictly positive price for fiat money exists (Proposition 3.1).

Once existence is established, the properties of the rationing equilibrium remain to be studied. In parallel papers Florig and Rivera (2005a, 2005b)), we establish a First and Second Welfare theorem and core equivalence for our equilibrium concept. Moreover, we study the behavior of rationing equilibria when the level of indivisibility converges to zero. Under suitable conditions a rationing equilibrium converges to a Walrasian equilibrium or otherwise to a hierarchic equilibrium (Florig (2001)).

So far, we have not mentioned the vast literature on indivisible goods. One could roughly divide it into two different approaches. The first, following Shapley and Scarf (1974) model markets without perfectly divisibles goods and consider only one commodity per agent, e.g. houses. The second of which, following Henry (1970), numerous authors (including Broome (1972), Mas-Colell (1977), Kahn and Yamazaki (1981), Quinzii (1984); see Bobzin (1998) for a survey) consider economies with indivisible commodities and one perfectly divisible commodity called *money*. However, this should not be confused with fiat money since it is a crucial consumption good. All these contributions suppose that the divisible commodity satisfies overriding desirability, i.e. it is so desirable to the agents that it can replace the consumption of indivisible goods. Moreover, every agent initially owns an important quantity of this good in the sense that no bundle of indivisible goods can yield as much utility as consuming his initial endowment of the divisible good and nothing of the indivisible one. Then, non-emptiness of the core and existence of a Walrasian equilibrium can be established.

The approach that is closest to this paper is of Dierker (1971) who established the existence of a quasi-equilibrium for exchange economies without a perfectly divisible consumption good. However, at a quasi-equilibrium, agents do not necessarily receive an individually rational commodity bundle - a drawback that is overcome by the rationing equilibrium.

2 Motivating examples

Before entering in specific details of the model, we go through some examples that motivate the definitions we will introduce in next sections.

(i) *Fiat money may change the set of Walrasian equilibria.*

Let $I = \{1, 2, 3\}$ be the set of consumers, $u_i(x, y) = x \cdot y$ the utility function for individual $i \in I$ and $e_1 = (7, 0), e_2 = (0, 3), e_3 = (0, 4) \in \mathbb{R}^2$ the respective initial endowment for them. Given $n \in \mathbb{N},\ n > 7$, for all $i \in I$ denote by $X_i = \{0, \ldots, n\}^2$ the individual consumption set. In this case, there exists a unique Walrasian equilibrium price $p = (1, 1)$ with the equilibrium allocations $x_1 = (4, 3), x_2 = (1, 2), x_3 = (2, 2)$ and $x'_1 = (3, 4),\ x'_2 = (2, 1),\ x'_3 = (2, 2)$. Suppose we now introduce fiat money into the economy such that each consumer can hold any positive amount of it and that fiat money does not enter in consumers' preferences. The new consumption sets are $X_i \times \mathbb{R}_+$ and if we endow each consumer with an initial amount of fiat money given by $0 < m_1 < 1/8$, $m_2 = 1$ and $0 < m_3 < 1/2$, it is easy to check that $p^* = (1, \frac{9}{8})$, $q^* = 1$, $x^*_1 = (3, 3)$, $x^*_2 = (2, 2)$ and $x^*_3 = (2, 2)$ is a new Walrasian equilibrium (with money) for this economy. The "consumption" of fiat money is determined by the Walrasian law, and for individual $i \in I$ is equal to

$$(p^* \cdot e_i + q^* m_i - p^* \cdot x^*_i)/q^* \geq 0.$$

(ii) *Without fiat money markets may be non viable*

Consider an exchange economy with one good, two consumers $I = \{1, 2\}$, consumption sets $X_i = \{0, 1, 2\}$ for $i \in I$, and utility functions $u_1(x) = -x$, $u_2(x) = x$. Finally, let $e_1 = 2$ and $e_2 = 0$ be the initial endowments. In this economy there is no Walrasian equilibrium: if $p \leq 0$ then Walrasian demand will be above the total initial endowment of the economy; if $p > 0$, the total initial endowment is above demand. However, if we endow consumer 2 with an initial amount of fiat money $m_2 > 0$, then prices $p = m_2,\ q = 1$, and demands for each type of individual given by $x_1 = 0,\ x_2 = 1$, is a Walrasian equilibrium.

(iii) *The rationing equilibrium*

Consider an exchange economy with three consumers indexed by $I = \{1, 2, 3\}$ and two goods. For all $i \in I$, let $X_i = \{0, 1, 2, 3\}^2$ be the respective individual consumption sets. Let $u_1(x, y) = x + 2y$ and $u_2(x, y) = u_3(x, y) = 2x + y$ be the utility functions for each consumer and let $e_1 = (1, 0), e_2 = (0, 1), e_3 = (0, 1) \in I\!R^2$ be the initial endowment for them. Given that, it is possible to show that no Walrasian equilibrium exists in the economy: the only price at which individual exchanges (of zero net-worth) can take place is $p = (1, 1)$, and in such case consumers 2 and 3 prefer to consume a unit of good 1 rather than keeping their initial endowment.

Reasonably one might expect that such an economy will reach either the state $x_1 = (0, 1), x_2 = (1, 0), x_3 = (0, 1)$ or $y_1 = (0, 1), y_2 = (0, 1), y_3 = (1, 0)$. If goods where perfectly divisible one may reject both allocations with an heuristic argument as the following: at both allocations the goods are exchanged at price $(1, 1)$, instead of three consumers, think rather of three types of consumers with a continuum of identical consumers of each type. Surely no consumer can manipulate the market price under these conditions. However if a a non-negligible set of consumers of type 3 does not receive the allocation $(1, 0)$ then they could propose to exchange at the price $(1 + \varepsilon, 1)$ for $\varepsilon > 0$ (buying the bundle $(1/(1 + \varepsilon), 0)$) in order to be "served" first. This would force the market price to move and reject x or y as an "equilibrium" situation. With continuous preferences it should be only at a Walrasian equilibrium that consumers would have no incentive to propose a slightly better price than the market price in order to be served first.

If goods are indivisible as stated in the example, consumers of type 2 or 3 are not able to propose an exchange at a slightly better price than the market price, because it would result in an allocation outside their consumption set. So having such a process in mind, x and y should be possible equilibrium situations and at equilibrium there are consumers who do not receive a maximal element within their budget set. Note that the same holds if we introduce fiat money, say $m = m_1 = m_2 = m_3 > 0$. Then at the price $p = (1 + m, 1)$ and price for money $q = 1$, again x and y should be equilibrium allocations by the same argument. However, if $m_2 > m_3$ only x should be an equilibrium allocation. Indeed, let $p = (z, 1)$ and the price of money $q = 1$, then as long as $z < m_2$ consumer 2 (or consumers of type 2) does not receive the bundle $(1, 0)$ he can propose to exchange at the price $(z + \varepsilon, 1)$ with $\varepsilon \in]0, m_2 - z]$ in order to be served first. As long as $z < m_3$ consumer 1 can do the same, but as soon as $z \geq m_2$, consumer 3 cannot do this anymore, he has not enough money. So we would have a unique equilibrium allocation, x which would either be a Walrasian equilibrium, if $p = (z, 1)$ and $q = 1$ with $z \in]m_3, m_2]$ or an equilibrium with rationing if $p = (m_3, 1)$ and $q = 1$.

In order to formalize all the above, we will introduce the *rationing equilibrium* concept in Section 3.

3 The model

3.1 Basic concepts

We set $L \equiv \{1, \ldots, L\}$ to denote the finite set of commodities. Let $I \equiv \{1, \ldots, I\}$ and $J \equiv \{1, \ldots, J\}$ be finite sets of types of identical consumers and producers respectively.

We assume that each type $k \in I, J$ of agents consists of a continuum of identical individuals represented by a set $T_k \subset I\!R$ of finite Lebesgue measure[2]. We set $\mathcal{I} = \cup_{i \in I} T_i$ and $\mathcal{J} = \cup_{j \in J} T_j$. Of course, $T_k \cap T_{k'} = \emptyset$ if $k \neq k'$. Given $t \in \mathcal{I}$ $(\mathcal{J})$, let

$$i(t) \in I \ \ (j(t) \in J)$$

be the index such that $t \in T_{i(t)}$ $(t \in T_{j(t)})$.

Each firm of type $j \in J$ is characterized by a *finite* production set $Y_j \subset I\!R^L$ and the aggregate production set of firms of type j is the convex hull of $\mathcal{L}(T_j)Y_j$, which is denoted by

$$\mathrm{co}\,[\mathcal{L}(T_j)Y_j] = \left\{ \sum_{r=0}^{n} \lambda_r y_r \mid y_r \in \mathcal{L}(T_j)Y_j, \ \lambda_r \geq 0, \sum_{r=0}^{n} \lambda_r = 1, \ n \in I\!N \right\}.$$

Every consumer of type $i \in I$ is characterized by a *finite* consumption set $X_i \subset R^L$, an initial endowment $e_i \in I\!R^L$ and a *preference correspondence* $P_i : X_i \to X_i$. Let $e = \sum_{i \in I} \mathcal{L}(T_i) e_i$ be the aggregate initial endowment of the economy. For $(i, j) \in I \times J$, $\theta_{ij} \geq 0$ is the share of type i consumers in type j firms. For all $j \in J$, assume that $\sum_{i \in I} \mathcal{L}(T_i)\theta_{ij} = 1$.

The initial endowment of fiat money for an individual $t \in \mathcal{I}$ is defined by $m(t)$, where $m : \mathcal{I} \to I\!R_+$ is a Lebesgue-measurable and bounded mapping. Without loss of generality we may assume that $m(\cdot)$ is a continuous mapping.

In the rest of this work, we note by $\mathcal{L}^1(A, B)$ the Lebesgue integrable functions from $A \subset I\!R$ to $B \subset I\!R^L$.

Given all the above, an *economy* $\mathcal{E}$ is a collection

$$\mathcal{E} = \left((X_i, P_i, e_i, m)_{i \in I}, (Y_j)_{j \in J}, (\theta_{ij})_{(i,j) \in I \times J} \right),$$

an *allocation* (or consumption plan) is an element of

$$X = \left\{ x \in \mathcal{L}^1(\mathcal{I}, \cup_{i \in I} X_i) \,|\, x_t \in X_{i(t)} \text{ for a.e. } t \in \mathcal{I} \right\}$$

and a *production plan* is an element of

[2] Without loss of generality we may assume that T_k is a compact interval of $I\!R$. In the following, we note by $\mathcal{L}(T_k)$ the Lebesgue measure of set $T_k \subseteq I\!R$.

$$Y = \left\{ y \in \mathcal{L}^1(\mathcal{J}, \cup_{j \in J} Y_j) \,|\, y_t \in Y_{j(t)} \text{ for a.e. } t \in \mathcal{J} \right\}.$$

Finally, the *feasible consumption-production plans* are elements of

$$A(\mathcal{E}) = \left\{ (x, y) \in X \times Y \mid \int_{\mathcal{I}} x_t = \int_{\mathcal{J}} y_t + e \right\}.$$

3.2 Equilibrium concepts

Given $p \in I\!R^L_+$, let us define the *supply* and *profit* of a type $j \in J$ firm as

$$S_j(p) = \text{argmax}_{y \in Y_j} \; p \cdot y \qquad\qquad \pi_j(p) = \mathcal{L}(T_j)\text{sup}_{y \in Y_j} \; p \cdot y$$

and for $(p, q) \in I\!R^L \times I\!R_+$, we denote the *budget set* of a consumer $t \in \mathcal{I}$ by

$$B_t(p, q) = \left\{ x \in X_{i(t)} \,|\, p \cdot x \leq w_t(p, q) \right\}$$

where

$$w_t(p, q) = p \cdot e_{i(t)} + qm(t) + \sum_{j \in J} \theta_{i(t)j} \pi_j(p)$$

is the wealth of individual $t \in \mathcal{I}$. The set of *maximal elements* for the preference relation in the budgetary set for consumer $t \in \mathcal{I}$ is denoted by

$$d_t(p, q) = \left\{ x \in B_t(p, q) \,|\, B_t(p, q) \cap P_{i(t)}(x) = \emptyset \right\}.$$

A collection $(x, y, p, q) \in A(\mathcal{E}) \times I\!R^L \times I\!R_+$ is a *Walrasian equilibrium with fiat money* of $\mathcal{E}$ if

(i) for a.e. $t \in \mathcal{I}$, $x_t \in d_t(p, q)$,

(ii) for a.e. $t \in \mathcal{J}$, $y_t \in S_{j(t)}(p)$.

As already was stressed in the introduction, and illustrated in Section 2, a Walrasian equilibrium may fail to exist in the presence of indivisible goods. Technically the main reason comes from the lack of regularity of the demand correspondence in presence of indivisible goods. The following example illustrates this point: suppose that the preference relation of a certain individual is represented by the utility function $u(x, y) = 2x + y$, that his initial endowment is $e = (0, 1)$ and his consumption set is $X = \{0, 1\}^2$. Then, given $p^n = (1 + 1/n, 1) \to p = (1, 1)$, $q^n = 0 = q$, we obtain that $d(p^n, q^n) \to (0, 1)$, whereas $d(p, q) = (1, 0)$, which proves that $d(\cdot)$ is not upper semi-continuous at $p = (1, 1)$.

We will first introduce an auxiliary regularized notion of demand. For a consumer $t \in \mathcal{I}$, given $p \in I\!R^L$ and $q \in I\!R_+$ we define his *weak demand* by[3]

[3]See e.g. Rockafellar and Wets (1998), Section, 5 for the *limsup* definition of a correspondence. We recall that the *limsup* of a correspondence is, by definition, the smallest upper semi-continuous correspondence that contains the original one. In next Section we will give an economic interpretation of this demand.

$$D_t(p,q) = \limsup_{(p',q')\to(p,q)} d_t(p',q').$$

Finally, our main demand notion employs pointed cones, that is, the set $\mathcal{C}$ of closed convex cones $K \subset I\!R^L$ such that $-K \cap K = 0_{I\!R^L}$.

For $(p,q,K) \in I\!R^L \times I\!R_+ \times \mathcal{C}$ we define the *demand* of a consumer $t \in \mathcal{I}$ by

$$\delta_t(p,q,K) = \left\{ x \in D_t(p,q) \,|\, P_{i(t)}(x) - x \subset K \right\}$$

and the *supply* of a firm $t \in \mathcal{J}$ by

$$\sigma_t(p,K) = \left\{ y \in S_{j(t)}(p) \,|\, Y_{j(t)} - y \subset -K \right\}.$$

This new notion of demand will be discussed in the next section, where we will also clarify the role of the pointed cone on its definition.

Definition 3.1 *A collection* $(x,y,p,q,K) \in A(\mathcal{E}) \times I\!R^L \times I\!R_+ \times \mathcal{C}$ *is a* **rationing equilibrium** *of* $\mathcal{E}$ *if*

(i) for a.e. $t \in \mathcal{I}$, $x_t \in \delta_t(p,q,K)$,

(ii) for a.e. $t \in \mathcal{J}$, $y_t \in \sigma_t(p,K)$.

Note that a Walrasian equilibrium with fiat money is of course a rationing equilibrium[4]. Note that for $q > 0$, the *demand for money* of consumer $t \in \mathcal{I}$ is

$$\mu_t = \frac{1}{q}\left(p \cdot e_{i(t)} + qm(t) + \sum_{j\in J} \theta_{i(t)j}\pi_j(p) - p \cdot x_t \right).$$

Walrasian law implies that the money market is in equilibrium at a rationing equilibrium. In example (iii) in Section 2, if we replace each agent by a continuum of agents with Lebesgue measure 1 and setting $m_2(\cdot)$ and $m_3(\cdot)$ identically equal to 1, as we already stated a Walrasian equilibrium does not exist. However,

$$p = (1,1),\ q = 1,\ K = \{\mu(1,-1) + \lambda(-1,1.5)|\ (\mu,\lambda) \geq 0\} \subseteq I\!R^2$$

and allocations $x_1 = (0,1), x_2 = (1,0), x_3 = (0,1)$ and $y_1 = (0,1), y_2 = (0,1), y_3 = (1,0)$ (adapted to the present continuum of agents framework) would be the rationing equilibria for the economy. It is interesting to note that only the part of the cone K which is orthogonal to p is important, i.e. we could alternatively define

$$\delta_t(p,q,K) = \left\{ x \in D_t(p,q) \,|\, p^\perp \cap (P_{i(t)}(x) - x) \subset K \right\}$$

and

$$\sigma_t(p,K) = \left\{ y \in S_{j(t)}(p) \,|\, p^\perp \cap (Y_{j(t)} - y) \subset -K \right\},$$

[4] We refer to Kajii (1996) for the links among Walrasian equilibrium, Walrasian equilibrium with fiat money and the *dividend equilibrium* notion.

to obtain the same equilibrium allocations as before. This can in fact be deduced from Proposition 4.1 in the next section. In our example the cone K would then simply be the cone generated by $(1, -1) \in I\!R^2$.

The next proposition shows that a rationing equilibrium is a Walrasian equilibrium if money supply is in a *generic position*. This proposition, together with Theorem 5.1 give us the necessary conditions on the economy to ensure the existence of a Walrasian equilibrium in our setting (Corollary 5.1).

Proposition 3.1 *Let (x, y, p, q, K) be a rationing equilibrium with $q > 0$, suppose for all $M \in I\!R$*

$$\mathcal{L}(\{t \in \mathcal{I} \ \mid m(t) = M\}) = 0,$$

then (x, y, p, q) is a Walrasian equilibrium.

Proof. We proceed by contraposition. Since the money endowments are in a generic position, for almost every $t \in \mathcal{I}$, $m(t) > 0$. Thus, from Theorem 5.1 there exists a rationing equilibrium (x, y, p, q, K) with $q > 0$. If the conclusion were false, by the finiteness of the consumption sets, there would exist $i \in I$, $T_i' \subseteq T_i$ with $\mathcal{L}(T_i') > 0$ and $\xi \in X_i$ such that for all $t \in T_i'$, $\xi \in P_{i(t)}(x_t)$ and $p \cdot \xi \leq w_t(p, q)$. By Proposition 4.1 in the next section, for a.e. $t \in T_i'$, $p \cdot \xi \geq w_t(p, q)$. Thus for a.e. $t \in T_i'$, $p \cdot \xi = w_t(p, q)$ and since $q > 0$, for a.e. $t \in T_i'$, $m_i(t) = (p \cdot \xi - p \cdot e_i + \sum_{j \in J} \theta_{ij} \pi_j(p))/q)$. This means that $m_i(\cdot)$ is constant almost everywhere on T_i'. Since $\mathcal{L}(T_i') > 0$, this yields a contradiction. $\square$

4 Demand characterization and interpretation

We will first characterize the weak demand, which, as we have already mentioned, is a key ingredient in defining our notion of demand. The most important part is case (a), when the value of fiat money is strictly positive. Cases (b) and (c) are given for the sake of completeness. We recall that in absence of indivisible goods, the demand characterization we give coincides with the standard demand definition. The proof of the Proposition 4.1 is given in the Appendix.

Proposition 4.1 *Given $t \in \mathcal{I}$, we have that:*

(a) if $qm(t) > 0$ then

$$D_t(p, q) = \left\{ x \in B_t(p, q) \middle| \ p \cdot P_{i(t)}(x) \geq w_t(p, q), \ x \notin \mathrm{co} P_{i(t)}(x) \right\},$$

(b) if $m(t) > 0$ then

$$D_t(p, q) = \left\{ x \in B_t(p, q) \middle| \begin{array}{c} p \cdot P_{i(t)}(x) \geq w_t(p, q), \\ \mathrm{co} P_{i(t)}(x) \cap \mathrm{co}\{x, e_{i(t)} + \sum_{j \in J} \theta_{i(t)j} \mathcal{L}(T_j) Y_j\} = \emptyset \end{array} \right\},$$

(c) if $m(t) = 0$ *then*

$$D_t(p,q) = \Big\{x \in B_t(p,q) | p \cdot P_{i(t)}(x) \geq w_t(p,q),\ \mathrm{co}P_{i(t)}(x) \cap C(p,x) = \emptyset\Big\}$$

where

$$C(p,x) = \mathrm{co}\left\{\theta x + (1-\theta)\left[e_{i(t)} + \sum_{j \in J} \theta_{i(t)j} \mathcal{L}(T_j) \mathrm{argmax}\, \pi_j(p)\right] \middle| \theta \geq 0\right\}.$$

Note that the condition $x \notin \mathrm{co}P_{i(t)}(x)$ in Proposition 4.1 (a) is redundant if one considers the demand as defined for the rationing equilibrium.

The economic interpretation of our demand notion is the following. As we already know, the presence of indivisible goods implies that in our model a consumer $t \in \mathcal{I}$ might be unable to obtain a maximal element within his budget set. The question which arises is what should the appropriate competitive solution be. Should a consumer be unable to buy $\xi \in B_{i(t)}(p,q)$ with $p \cdot \xi < w_{i(t)}(p,q)$, then he could try to pay for this bundle at a higher price than the market price in order to be "served first". Thus, there is some pressure on the price of the bundle ξ and its price would rise, if a non-negligible set of consumers is rationed in this sense. So at equilibrium, no consumer obtains a bundle of goods $x \in B_{i(t)}(p,q)$ such that a strictly preferred bundle ξ with $p \cdot \xi < w_{i(t)}(p,q)$ exists. Thus, a competitive solution (x,y,p,q) should at least satisfy

$$p \cdot P_{i(t)}(x) \geq w_t(p,q)$$

for all i and almost all t. The last constraint however is not enough for a satisfying equilibrium notion. It would be very "fragile" with respect to the information agents have on other agents allocations and information. If net-trades would exist which strictly benefit all participating agents, when these exchanges might be carried out, and some exchange process might carry on. However, one does not need to know everything about every potential trading partner in order to reach a situation where this is not possible. It is enough to have an aggregate knowledge of the market, which summarizes which kind of net-trades are difficult to realize on the market (which is the "short" side of the market) when formulating their demand. This short side of the market could be modelled using a cone $K \subseteq I\!R^L$ which does not contain straight lines, i.e. if a direction of net-trade is difficult to realize, the opposite direction is easy to realize. This is precisely the role of a pointed cone K in the rationing equilibrium definition. This point is illustrated by the third example in section two, together with the following example: consider an exchange economy with three types of consumers $I = \{1,2,3\}$ and for each type there are a continuum of them, indexed by compact and disjoint intervals $T_i \subseteq I\!R, i \in I$, with identical Lebesgue measure. Suppose there are two commodities and for all $i \in I$, $X_i = \{0,1,2\}^2$, $u_1(x) = -x^1 - x^2$, $u_2(x) = 2x^1 + x^2$, $u_3(x) = x^1 + 2x^2$, $e_1 = (1,1), e_2 = e_3 = (0,0)$ (cf. Konovalov 2005). If $m_1 = m_2 = m_3 = 1$, then (x,p,q) such that all consumers have the same allocation, respectively $x_1 = (0,0)$, $x_2 = (0,1)$, $x_3 = (1,0)$ is feasible. Moreover, for $p = (1,1)$, $q = 1$ it satisfies $x(t) \in D_t(p,q)$ for all $t \in \mathcal{I}$. However, it is not a rationing equilibrium. If this allocation were to be realized, consumers of type two and three would wish to swap their allocations leading to $\xi_2 = (1, 0)$ for all type two consumers and $\xi_3 = (0, 1)$ for all type three consumers (and $\xi_1 = x_1 = (1, 0)$ for all type one consumers). So (ξ, p, q) is together with for example the convex cone generated by $(1, -1), (-1, 1.5)$ is a rationing equilibrium. Note also that (ξ, p, q) is a Walrasian equilibrium.

To end this Section, it is necessary to mention that the employment of a cone -more specifically, a closed convex set with zero as an interior point- as a set of information that restricts the possible trades of consumers has been also employed in a different setting by Hammond (2003) in order to study the existence of a Walrasian equilibrium without transfers. However, in his setting this "cone" is given ex ante as a part of the individual's endowments, whereas in our case its existence is ensured as a part of the equilibrium definition.

5 Existence of equilibrium

The strongest condition we use to ensure existence of equilibrium is the finiteness of the consumption and production sets. The rest of our assumptions are quite weak. In particular, we do not need a strong survival assumption, that is, our consumers may not own initially a strictly positive quantity of every good and the interior of the convex hull of the consumption sets may be empty (cf. Arrow and Debreu (1954)).

Assumption C. For all $i \in I$, P_i is irreflexive and transitive, that is, for each $i \in I$ and $x, y, z \in X_i$, $x \notin P_i(x)$ and if $x \in P_i(y)$ and $y \in P_i(z)$ then $x \in P_i(z)$.

Assumption S. (Weak survival assumption). For all $i \in I$,

$$0 \in \mathrm{co}X_i - \{e_i\} - \sum_{j \in J} \theta_{ij} \mathcal{L}(T_j) \mathrm{co}Y_j.$$

Theorem 5.1 *For every economy $\mathcal{E}$ satisfying Assumptions C, S and $m(t) > 0$ for a.e. $t \in \mathcal{I}$, there exists a rationing equilibrium with the price of fiat money being strictly positive.*

The following corollary states the existence of a Walrasian equilibrium, generically on the distribution of fiat money. It follows directly from Theorem 5.1 and Proposition 3.1.

Corollary 5.1 *For every economy $\mathcal{E}$ satisfying Assumptions C, S and for all $M \in I\!R$*

$$\mathcal{L}(\{t \in \mathcal{I} \ \mid m(t) = M\}) = 0,$$

then there exists a Walrasian equilibrium with the price of fiat money being strictly positive.

The proof of Theorem 5.1 will be done based on an induction argument using an auxiliary equilibrium notion. A collection $(x, y, p, q) \in A(\mathcal{E}) \times I\!R^L \times I\!R_+$ is a *weak equilibrium* of $\mathcal{E}$ if

(i) for a.e. $t \in \mathcal{I}$, $x_t \in D_t(p, q)$,

(ii) for a.e. $t \in \mathcal{J}$, $y_t \in S_{j(t)}(p)$.

A rationing equilibrium is by definition a weak equilibrium. The following Lemma establishes existence of the weak equilibrium, which is an intermediary step in proving existence of a rationing equilibrium. The Lemma will be proven in the Appendix.

Lemma 5.1 *For every economy $\mathcal{E}$ satisfying Assumptions C, S, there exists a weak equilibrium with the price of fiat money being strictly positive.*

6 Conclusion

In an economic framework where all goods are indivisible at the individual level but perfectly divisible at the aggregate level, we introduced a new competitive equilibrium notion called rationing equilibrium (Definition 3.1). We proved its existence under rather general assumptions on the economy (Theorem 5.1) and, when the initial endowments of fiat money are in a generic position, we proved that the rationing equilibrium is a Walrasian one (Proposition 3.1). As a byproduct, we give another approach where fiat money plays an essential role in the economy: precisely its role as a parameter to facilitate exchange when all goods are indivisible implies that its price at the equilibrium must be strictly positive. In two parallel papers we study welfare properties, core equivalence and the properties of the economy when the level of indivisibility becomes small (Florig and Rivera (2005a, 2005b)).

7 Appendix

7.1 Proof of Proposition 4.1.

Part (a). Given $t \in \mathcal{I}$, let

$$a(p,q) = \left\{ x \in B_t(p,q) \mid p \cdot P_{i(t)}(x) \geq w_t(p,q),\ x \notin \mathrm{co}P_{i(t)}(x) \right\}.$$

First of all, note that by definition $D_t(p,q) \subset a(p,q)$. Let $x \in a(p,q)$. If $p \cdot x < w_t(p,q)$, then for all small enough $\varepsilon > 0$, $x \in d_t(p, q-\varepsilon)$ and hence $x \in D_t(p,q)$. Otherwise, note that there exists p' such that $p' \cdot P_{i(t)}(x) > p' \cdot x$. For all $\varepsilon > 0$, let $p^\varepsilon = p + \varepsilon p'$ and let

$$q^\varepsilon = \left[\frac{p^\varepsilon \cdot (x - e_{i(t)}) - \sum_{j \in J} \theta_{i(t)j} \pi_j(p^\varepsilon)}{m(t)} \right].$$

Note that $\lim_{\varepsilon \to 0}(p^\varepsilon, q^\varepsilon) = (p,q)$. Moreover for all $\varepsilon > 0$,

$$p^\varepsilon \cdot P_{i(t)}(x) > p^\varepsilon \cdot x = w_t(p^\varepsilon, q^\varepsilon).$$

Since for $\varepsilon > 0$ small enough, $q^\varepsilon > 0$, we have $x \in D_t(p,q)$. Thus $a(p,q) \subset D_t(p,q)$.

Part (b). Let

$$A(p,q) = \left\{ x \in B_t(p,q) \,\middle|\, \begin{array}{c} p \cdot P_{i(t)}(x) \geq w_t(p,q), \\ \mathrm{co}P_{i(t)}(x) \cap \mathrm{co}\{x, e_{i(t)} + \sum_{j \in J} \theta_{i(t)j} \mathcal{L}(T_j) Y_j\} = \emptyset \end{array} \right\}.$$

Step b.1. $A(p,q) \subset D_t(p,q)$.

Let $x \in A(p,q)$. Thus, there exists p' such that

$$p' \cdot P_{i(t)}(x) > p' \cdot \left\{ x, e_{i(t)} + \sum_{j \in J} \theta_{i(t)j} \mathcal{L}(T_j) Y_j \right\}.$$

For all $\varepsilon > 0$, let $p^\varepsilon = p + \varepsilon p'$. Thus, for all $\varepsilon > 0$,

$$p^\varepsilon \cdot P_{i(t)}(x) > p^\varepsilon \cdot \left\{ x, e_{i(t)} + \sum_{j \in J} \theta_{i(t)j} \mathcal{L}(T_j) Y_j \right\},$$

$$p^\varepsilon \cdot P_{i(t)}(x) > w_t(p^\varepsilon, q).$$

Let[5]

$$q^\varepsilon = q + \left[\frac{p^\varepsilon \cdot x - w_t(p^\varepsilon, q)}{m(t)} \right]_+ .$$

Note that $\lim_{\varepsilon \to 0}(p^\varepsilon, q^\varepsilon) = (p,q)$ and moreover for all $\varepsilon > 0$,

$$p^\varepsilon \cdot P_{i(t)}(x) > w_t(p^\varepsilon, q^\varepsilon) \geq p^\varepsilon \cdot x$$

and therefore $x \in D_t(p,q)$.

Step b.2. $D_t(p,q) \subset A(p,q)$.

For all $x \in D_t(p,q)$, there exists sequences (p^n, q^n) converging to (p,q), such that for all $n \in I\!N$

$$p^n \cdot P_{i(t)}(x) > w_t(p^n, q^n) \geq p^n \cdot x.$$

Thus $p \cdot P_{i(t)}(x) \geq w_t(p,q)$ and

$$\mathrm{co} P_{i(t)}(x) \cap \mathrm{co} \left\{ x, e_{i(t)} + \sum_{j \in J} \theta_{i(t)j} \mathcal{L}(T_j) Y_j \right\} = \emptyset$$

which ends the proof of part (b).

Part (c).

Let

$$c(p) = \left\{ x \in B_t(p,q) \;\middle|\; \begin{array}{c} p \cdot P_{i(t)}(x) \geq w_t(p,q), \\ \mathrm{co} P_{i(t)}(x) \cap C(p,x) = \emptyset \end{array} \right\}.$$

Step c.1. $c(p) \subset D_t(p,q)$.

Given $x \in c(p)$ there exists p' such that

$$p' \cdot \mathrm{co} P_{i(t)}(x) > p' \cdot \left(e_{i(t)} + \sum_{j \in J} \theta_{i(t)j} \mathcal{L}(T_j) \operatorname{argmax} \pi_j(p) \right) \geq p' \cdot x.$$

[5] For $x \in I\!R$, we note $[x]_+ = \max\{x, 0\}$.

Thus, for all $\varepsilon > 0$, given $p^\varepsilon = p + \varepsilon p'$ it follows that

$$\min p^\varepsilon \cdot P_{i(t)}(x) > \max p^\varepsilon \cdot \left(e_{i(t)} + \sum_{j\in J} \theta_{i(t)j} \mathcal{L}(T_j) \mathrm{argmax}\, \pi_j(p)\right),$$

$$\min p^\varepsilon \cdot \left(e_{i(t)} + \sum_{j\in J} \theta_{i(t)j} \mathcal{L}(T_j) \mathrm{argmax}\, \pi_j(p)\right) \geq p^\varepsilon \cdot x.$$

Moreover, since Y_j is finite for all $j \in J$, we may check that for all $\varepsilon > 0$ small enough and all $j \in J$,

$$\mathrm{argmax}\, \pi_j(p^\varepsilon) \subset \mathrm{argmax}\, \pi_j(p)$$

and therefore for all small $\varepsilon > 0$,

$$\min p^\varepsilon \cdot P_{i(t)}(x) > w_t(p^\varepsilon, q) \geq p^\varepsilon \cdot x,$$

which implies that $x \in D_t(p, q)$.

Step c.2. $D_t(p, q) \subset c(p)$.

Let $x \in D_t(p, q)$. Then there exists a sequence p^n converging to p such that for all $n \in I\!N$,

$$p^n \cdot P_{i(t)}(x) > w_t(p^n, q) \geq p^n \cdot x.$$

Thus $p \cdot P_{i(t)}(x) \geq w_t(p, q)$ and p^n separates strictly $\mathrm{co}P_{i(t)}(x)$ and

$$\mathrm{co}\left\{\theta x + (1-\theta)[e_{i(t)} + \sum_{j\in J} \theta_{i(t)j} \mathcal{L}(T_j) Y_j] \mid \theta \geq 0\right\}.$$

Since

$$C(p, x) \subset \mathrm{co}\left\{\theta x + (1-\theta)[e_{i(t)} + \sum_{j\in J} \theta_{i(t)j} \mathcal{L}(T_j) Y_j] \mid \theta \geq 0\right\}$$

we can conclude that $x \in c(p)$. □

7.2 Proof of Lemma 5.1

In order to demonstrate Lemma 5.1 we use the following proposition, which is an extension of the well know Debreu-Gale-Nikaido lemma.

Proposition 7.1 *Let $\varepsilon \in]0, 1]$ and φ be an upper semi continuous correspondence from $I\!B(0, \varepsilon)$ to $I\!R^L$ with nonempty, convex, compact values*[6]*. If for some $k > 0$,*

$$\forall p' \in I\!B(0, \varepsilon), \quad \|p'\| = \varepsilon \quad \Longrightarrow \quad \sup p' \cdot \varphi(p') \leq k(1-\varepsilon),$$

then there exists $p \in I\!B(0, \varepsilon)$ such that, either:

- $0 \in \varphi(p)$

or

- *$\|p\| = \varepsilon$ and $\exists\ \xi \in \varphi(p)$ such that ξ and p are colinear and $\|\xi\| \leq k\frac{1-\varepsilon}{\varepsilon}$.*

[6] $I\!B(0, \varepsilon) = \{x \in I\!R^L \mid \|x\| \leq \varepsilon\}$. The norm used here is Euclidean norm.

Proof of Proposition 7.1.

From the properties of φ, one can select a convex compact subset $K \subset I\!R^L$ such that for all $p \in I\!B(0,\varepsilon)$, $\varphi(p) \subset K$. Consider the correspondence $F : I\!B(0,\varepsilon) \times K \to I\!B(0,\varepsilon) \times K$ defined by

$$F(p,z) = \{q \in I\!B(0,\varepsilon) \mid \forall q' \in I\!B(0,\varepsilon),\ q \cdot z \geq q' \cdot z\} \times \varphi(p).$$

From Kakutani Theorem, F has a fixed point (p,ξ). If $\|p\| < \varepsilon$, then $\xi = 0$. If $\|p\| = \varepsilon$, then from the definition of F, p and ξ are colinear. Therefore, $\|\xi\| \leq k\ \frac{1-\varepsilon}{\varepsilon}$, which ends the demonstration. $\square$

Proof of Lemma 5.1

Previous to proceed, is necessary to introduce some notations. We note by $\leq_{lex}$ the lexicographic order[7]. Given $p_0, ..., p_k \in I\!R^L$, for a $(k+1) \times L$ matrix $\mathcal{P} = [p_0, \ldots, p_k]'$ (transpose of matrix $[p_0, \ldots, p_k]$), we note for every $j \in J$,

$$S_j(\mathcal{P}) = \{y \in Y_j \mid \forall z \in Y_j,\ \mathcal{P}z \leq_{lex} \mathcal{P}y\} \qquad\qquad \pi_j(\mathcal{P}) = \mathcal{L}(T_j)\mathrm{sup}_{lex}\{\mathcal{P}y \mid y \in Y_j\}$$

where sup_{lex} is the supremum with respect to the lexicographic order. Given $\mathcal{Q} = (q_r) \in I\!R^{k+1}$, for every $t \in \mathcal{I}$ let

$$B_t(\mathcal{P},\mathcal{Q}) = \left\{ x \in X_{i(t)} \mid \mathcal{P} \cdot (x - e_{i(t)}) \leq_{lex} m(t)\mathcal{Q} + \sum_{j \in J} \theta_{i(t)j}\pi_j(\mathcal{P}) \right\}$$

and finally, for $\varepsilon > 0$, we note $\mathcal{P}(\varepsilon) = \sum_{r=0}^{k} \varepsilon^r p_r$ and $\mathcal{Q}(\varepsilon) = \sum_{r=0}^{k} \varepsilon^r q_r$. So now we are in conditions to demonstrate the result. To do so, we proceed in nine steps.

Step 1. *Perturbed equilibria.*

For simplicity, for all $t \in \mathcal{I}$ we note $D_t(p)$ instead of $D_t(p, 1 - \|p\|)$. Given that, it is easy to check that for all $\varepsilon \in [0,1]$, all $t \in \mathcal{I}$, and all $j \in J$ the set-valued mappings

$$D_t : I\!B(0,\varepsilon) \to \mathrm{co}X_{i(t)} \qquad \mathrm{co}S_j : I\!B(0,\varepsilon) \to \mathrm{co}Y_j$$

are upper semi-continuous, nonempty and compact valued.

Now, define the excess demand mapping

$$\varphi : I\!B\,(0, 1 - 1/n) \to \sum_{i \in I} \mathcal{L}(T_i)(\mathrm{co}X_i - e_i) - \sum_{j \in J} \mathrm{co}\,[\mathcal{L}(T_j)Y_j]$$

by

$$\varphi(p) = \int_{t \in \mathcal{I}} (D_t(p) - e_{i(t)}) - \sum_{j \in J} \mathrm{co}\,[\mathcal{L}(T_j)S_j(p)]\,.$$

Obviously $\varphi(\cdot)$ is nonempty, convex, compact valued and upper semi-continuous. For each $n \in I\!N$ and each $p \in I\!B(0, 1 - 1/n)$ we have that

[7]For $(x,y) \in I\!R^n \times I\!R^n$, $x \leq_{lex} y$, if $x_r > y_r$, $r \in \{1, \ldots, n\}$ implies that $\exists \rho \in \{1, \ldots, r-1\}$ such that $x_\rho < y_\rho$. We write $x <_{lex} y$ if $x \leq_{lex} y$, but not $[y \leq_{lex} x]$. In an obvious manner we define $x \geq_{lex} y$ and $x >_{lex} y$.

$$p \cdot \varphi(p) \leq (1 - \|p\|) \int_{\mathcal{I}} m(t).$$

So we may apply Proposition 7.1 to conclude that for all $n > 1$ there exists

$$(x^n, y^n, p^n, q^n) \in \prod_{i \in I} L^1(T_i, X_i) \times \prod_{j \in J} L^1(T_j, S_j(p^n)) \times I\!\!B\,(0, 1 - 1/n) \times I\!\!R_{++}$$

such that for all $t \in \mathcal{I}$, $x_t^n \in D_t(p^n, q^n)$, $q^n = 1 - \|p^n\|$, $\int_{t \in \mathcal{I}} x_t^n + \int_{t \in \mathcal{J}} y_t^n - e \in \varphi(p^n)$ and

$$\left\| \int_{t \in \mathcal{I}} x_t^n + \int_{t \in \mathcal{J}} y_t^n - e \right\| \leq \frac{1}{n-1} \int_{\mathcal{I}} m(t).$$

Step 2. *Construction of $\mathcal{P}$ and $\mathcal{Q}$.*

For the construction of a hierarchic price we will proceed as in Florig (2002). For that, our objective is to define a set of vectors $\{\psi_0, \psi_1, ..., \psi_L\} \subseteq I\!\!R^{L+1}$ which help us to define both $\mathcal{P}$ and $\mathcal{Q}$ as required. To do so, set $\psi^n = (p^n, q^n)$ and taking a subsequence, we may assume that ψ^n converges to $(p_0, q_0) \in I\!\!R^{L+1}$. Let ψ_0, ψ_0^n and $\mathcal{H}^0$ defined as follows:

$$\psi_0 = (p_0, q_0),$$
$$\psi_0^n = \psi^n,$$
$$\mathcal{H}^0 = \psi_0^{\perp} = \{x \in R^{L+1} \mid \psi_0 \cdot x = 0\}.$$

Using a recursive procedure, for every $r \in \{1, 2, \ldots, L-1\}$ we define ψ_r, ψ_r^n and $\mathcal{H}^r$ as follows:

$$\psi_r^n = \mathrm{proj}_{\mathcal{H}^{r-1}}(\psi_{r-1}^n),$$

and given that, if for all large enough $n \in I\!\!N$, $\psi_r^n \neq 0$, then let $\psi_r \equiv (p_r, q_r)$ be the limit of $\psi_r^n / \parallel \psi_r^n \parallel$ for some subsequence. In such case,

$$\mathcal{H}^r = \psi_r^{\perp}$$

and

$$\psi_{r+1}^n = \mathrm{proj}_{\mathcal{H}^r}(\psi_r^n).$$

We continue with previous algorithm until for all large enough $n \in I\!\!N$, $\psi_r^n = 0$ for some subsequence. In such case, we set $\psi_r = \ldots = \psi_L = 0$ and define

$$k = \min\{r \in \{0, \ldots, L\} \mid \psi_{r+1} = \ldots = \psi_L = 0\}.$$

Given all foregoing, we had obtained a set $\{\psi_r = (p_r, q_r),\ r = 1, \ldots, k\}$ of orthonormal vectors. Note that for all $r \in \{0, \ldots, k\}$[8],

$$\left\| \psi_{r+1}^n \right\| = \|\psi_r^n\| \, o(\|\psi_r^n\|)$$

[8]Throughout the paper we denote by $o : R \to R$ a function which is continuous in 0 with $o(0) = 0$.

which allow us to decompose the sequence ψ^n in the following way

$$\psi^n = \sum_{r=0}^{k} (\| \psi_r^n \| - \| \psi_{r+1}^n \|)\psi_r = \sum_{r=0}^{k} \varepsilon_r^n \psi_r,$$

with $\varepsilon_r^n = \| \psi_r^n \| - \| \psi_{r+1}^n \|$ for $r \in \{0,\dots,k\}$. Thus, $\varepsilon_{r+1}^n = \varepsilon_r^n o(\varepsilon_r^n)$ for $r \in \{0,\dots,k-1\}$, and ε_0^n converges to 1.

Let $\mathcal{P} = [p_0,\dots,p_k]'$ (transpose of matrix $[p_0,\dots,p_k]$), and $\mathcal{Q} = (q_0, q_1, ..., q_k) \in \mathbb{R}^{k+1}$.

Step 3. *Equilibrium allocation candidate.*

There exists by Fatou's lemma (Arstein (1979)) $(x^*, y^*) \in A(\mathcal{E})$ such that for a.e. $t \in \mathcal{I}$ and a.e. $t' \in \mathcal{J}$[9]

$$x_t^* \in \mathrm{cl}\{x_t^n\}, \quad y_{t'}^* \in \mathrm{cl}\{y_{t'}^n\}.$$

Step 4. *For all $\varepsilon > 0$ small enough and all n large enough, for a.e. $t \in \mathcal{J}$,*

$$y_t^* \in S_{j(t)}(\mathcal{P}(\varepsilon)) = S_{j(t)}(p^n) = S_{j(t)}(\mathcal{P}).$$

Since for all $j \in J$, Y_j is finite, for all $\varepsilon > 0$ small enough and for all $j \in J$ we have that $S_j(\mathcal{P}(\varepsilon)) = S_j(\mathcal{P})$ and similarly, for $n \in \mathbb{N}$ large enough, for all $j \in J$, $S_j(p^n) = S_j(\mathcal{P})$. Since for a.e. $t \in \mathcal{J}$, $y_t^n \in S_{j(t)}(p^n)$ for all $n \in \mathbb{N}$, and since $y_t^* \in \mathrm{cl}\{y_t^n\}$, y_t^n is constant and equal to y_t^* for a subsequence. Thus, $y_t^* \in S_{j(t)}(\mathcal{P})$.

Let ρ be the smallest $r \in \{0,\dots,k\}$ such that $q_r \neq 0$. Since for all $n \in \mathbb{N}$, $q^n > 0$, $q_\rho > 0$. Let $\widetilde{\mathcal{P}} = [p_0,\dots,p_\rho]'$ and $\widetilde{\mathcal{Q}} = (q_0,\dots,q_\rho)$. For all $j \in J$, let $\bar{y}_j = y_t^*$, provided that $y_t^* \in S_j(\mathcal{P})$. Since that $S_j(\mathcal{P}) \subset S_j(\widetilde{\mathcal{P}})$, $\bar{y}_j \in S_j(\widetilde{\mathcal{P}})$.

Step 5. *For a.e. $t \in \mathcal{I}$, $x_t^* \in B_t(\widetilde{\mathcal{P}}, \widetilde{\mathcal{Q}})$.*

By the previous step, one may check that $B_t(p^n, q^n)$ converges in the sense of Kuratowski - Painlevé to $B_t(\mathcal{P}, \mathcal{Q})$[10]. Thus $x_t^* \in B_t(\mathcal{P}, \mathcal{Q}) \subset B_t(\widetilde{\mathcal{P}}, \widetilde{\mathcal{Q}})$.

Step 6. *For all $\varepsilon > 0$ small enough, for a.e. $t \in \mathcal{I}$, $x_t^* \in B_t(\widetilde{\mathcal{P}}(\varepsilon), \widetilde{Q}(\varepsilon))$.*

For a.e. $t \in \mathcal{I}$, we have, by the previous step, that $x_t^* \in B_t(\widetilde{\mathcal{P}}(\varepsilon), \widetilde{\mathcal{Q}}(\varepsilon))$ for all small enough $\varepsilon > 0$. Since $m : \mathcal{I} \to \mathbb{R}_+$ is bounded and since there are only finitely many values for x_t^*, there exists $\varepsilon > 0$ satisfying this property for a.e. $t \in \mathcal{I}$.

Step 7. *For a.e. $t \in \mathcal{I}$, $x_t \in P_{i(t)}(x_t^*)$ implies that*

$$\widetilde{\mathcal{P}} \cdot (x_t - e_{i(t)}) - m(t)\widetilde{\mathcal{Q}} - \sum_{j \in J} \theta_{i(t)j} \pi_j(\widetilde{\mathcal{P}}) \geq_{lex} 0.$$

Otherwise

$$\widetilde{\mathcal{P}} \cdot (x_t - e_{i(t)}) - m(t)\widetilde{\mathcal{Q}} - \sum_{j \in J} \theta_{i(t)j} \mathcal{L}(T_j)\widetilde{\mathcal{P}}\bar{y}_j <_{lex} 0$$

[9]In the following, the closure of set A is denoted by $\mathrm{cl}A$.

[10]This concept is widely used to define set - convergence. See Rockafellar and Wets (1998), Section 4, for more details.

and then for all large enough $n \in I\!N$,

$$p^n \cdot (x_t - e_{i(t)}) - q^n m(t) - \sum_{j \in J} \theta_{i(t)j} \mathcal{L}(T_j) p^n \cdot \bar{y}_j < 0.$$

By Proposition 4.1, this contradicts $x_t^* \in D_t(p^n, q^n)$ for a subsequence of (p^n, q^n).

Step 8. *For all $\varepsilon > 0$ small enough, for a.e $t \in \mathcal{I}$, $x_t \in P_{i(t)}(x_t^*)$ implies that*

$$\widetilde{\mathcal{P}}(\varepsilon) \cdot (x_t - e_{i(t)}) - \widetilde{\mathcal{Q}}(\varepsilon) m(t) - \sum_{j \in J} \theta_{i(t)j} \pi_j(\widetilde{\mathcal{P}}(\varepsilon)) \geq 0.$$

Since X_i is finite, there exists a finite partition $\{\widetilde{T}_1, \ldots, \widetilde{T}_f\}$ of $\mathcal{I}$ such that the sets $B_t(\widetilde{\mathcal{P}}, \widetilde{\mathcal{Q}})$ are constant on each of the elements of the partition. We may choose the partition such that for every $s \in \{1, \ldots, f\}$, there exists $i \in I$ such that $\widetilde{T}_s \subset T_i$ and x_t^* is constant on $\widetilde{T}_s$. Let $m^s = \text{essup}\{m(t) \mid t \in \widetilde{T}_s\}$ (essential supremum) and suppose for all $\bar{\varepsilon} > 0$, there exists $\varepsilon \in]0, \bar{\varepsilon}]$ such that

$$\widetilde{\mathcal{P}}(\varepsilon) \cdot (x_t - e_{i(t)}) - m^s \widetilde{\mathcal{Q}}(\varepsilon) - \sum_{j \in J} \theta_{i(t)j} \mathcal{L}(T_j) \widetilde{\mathcal{P}}(\varepsilon) \cdot \bar{y}_j < 0.$$

Thus there exists $\eta \in]0, m^s]$ such that for all large $n \in I\!N$,

$$p^n \cdot (x_t - e_{i(t)}) - q^n (m^s - \eta) - \sum_{j \in J} \theta_{i(t)j} \mathcal{L}(T_j) p^n \cdot \bar{y}_j < 0.$$

Hence, for all large $n \in I\!N$ there exists $\bar{T}_s \subset \widetilde{T}_s$ with $\mathcal{L}(\bar{T}_s) > 0$ such that for a.e. $t \in \bar{T}_s$

$$p^n \cdot (x_t - e_{i(t)}) - q^n m(t) - \sum_{j \in J} \theta_{i(t)j} \mathcal{L}(T_j) p^n \cdot \bar{y}_j < 0.$$

By Proposition 4.1, this contradicts $x_t^* \in D_t(p^n, q^n)$ for a subsequence of (p^n, q^n).

Step 9. *For all $\varepsilon > 0$ small enough, for a.e. $t \in \mathcal{I}$, $x_t^* \in D_t((\mathcal{P}(\varepsilon), (\mathcal{Q}(\varepsilon))$.*

Let $\bar{\varepsilon} > 0$ small enough satisfying the previous steps. Let $(p^*, q^*) = \sum_{r=0}^{\rho} \bar{\varepsilon}^r (p_r, q_r)$.

By Proposition 4.1, $x_t^* \notin \text{co}P_{i(t)}(x_t^*)$. Then, since $q^* > 0$ and for a.e. $t \in \mathcal{I}$, $m(t) > 0$, we can deduce by Proposition 4.1 that $x_t^* \in D_t((\mathcal{P}(\varepsilon), (\mathcal{Q}(\varepsilon))$.

Thus, (x^*, y^*, p^*, q^*) is a weak equilibrium and $q^* > 0$. □

7.3 Proof of Theorem 5.1

Let $m^1 : \mathcal{I} \to I\!R_{++}$ be a mapping strictly increasing and bounded and let (x^0, y^0, p^0, q^0) be a weak equilibrium of $\mathcal{E}$. Let $\mathcal{E}^1$ be an economy defined as follows. Since the number of types is finite and the consumption sets are finite, we can define a finite set of consumer types $A \equiv \{1, \ldots, A\}$ satisfying the following:

(i) $(T_a)_{a \in A}$ is a finer partition of $\mathcal{I}$ than $(T_i)_{i \in I}$,

(ii) for every $a \in A$, there exists x_a such that for every $t \in T_a$, $x_t^0 = x_a$.

Set $X_a^1 = (P_a(x_a) \cup x_a) \cap (x_a + (p^0)^\perp)$ and $e_a^1 = x_a$, with P_a^1 the restriction of P_a to X_a^1.

Since there is also a finite number of types of producers and production sets are finite, we can define a finite set of producer types $B \equiv \{1, \ldots, B\}$ satisfying the following:

(i) $(T_b)_{b \in B}$ is a finer partition of $\mathcal{J}$ than $(T_j)_{j \in J}$,

(ii) for every $b \in B$, there exists y_b such that for every $t \in T_b$, $y_t^0 = y_b$.

Given $Y_b^1 = ((Y_b - y_b) \cap (p^0)^\perp)$, define the economy by $\mathcal{E}^1$ as

$$\mathcal{E}^1 = \left((X_a^1, P_a^1, e_a^1, m^1)_{a \in A}, (Y_b^1)_{b \in B}, (\theta_{ab})_{(a,b) \in A \times B}\right),$$

where m^1 defines the initial endowments of fiat money. The economy $\mathcal{E}^1$ satisfies Assumptions C, S. So by the Lemma 5.1 there exists a weak equilibrium with $q^1 > 0$ and therefore a Walrasian equilibrium (with fiat money) for the economy $\mathcal{E}^1$, which is denoted by (x^1, y^1, p^1, q^1), with $q^1 > 0$. Set $\mathcal{P} = [p^0, p^1]'$.

Claim 7.1 *For a.e. $t \in \mathcal{I}$, $\mathcal{P}x_t^1 \leq_{lex} w_t$ with $w_t = (w_t^0, w_t^1) \in I\!R^2$ such that*

$$w_t^0 = p^0 \cdot e_{i(t)} + q^0 m(t) + \sum_{j \in J} \theta_{i(t)j} \mathcal{L}(T_j) p^0 \cdot y_j^0$$

$$w_t^1 = p^1 \cdot e_{i(t)}^1 + q^1 m^1(t) + \sum_{b \in B} \theta_{i(t)b} \mathcal{L}(T_b) p^1 \cdot y_b^1.$$

Note that by the construction of X_t^1, we have for a.e. $t \in \mathcal{I}$, $p^0 \cdot x_t^0 = p^0 \cdot x_t^1$. Since for every $r \in \{0, 1\}$, $p^r \cdot x_t^r \leq w_t^r$ we have for a.e. $t \in \mathcal{I}$, $\mathcal{P}x_t^1 \leq_{lex} w_t$. By other hand, note that for all $t \in \mathcal{J}$, $y_t = y_t^0 + y_t^1 \in S_{j(t)}(\mathcal{P})$.

Claim 7.2 *For a.e. $t \in \mathcal{I}$, $\xi_t \in P_{i(t)}(x_t^1)$ implies $\mathcal{P}x_t^1 <_{lex} \mathcal{P}\xi_t$.*

By transitivity of the preferences, $\xi_t \in P_{i(t)}(x_t^1)$ implies that $\xi_t \in P_{i(t)}(x_t^0)$. Thus, $p^0 \cdot x_t^1 = p^0 \cdot x_t^0 \leq p^1 \cdot \xi_t$. Since (x^1, y^1, p^1, q^1) is a Walrasian equilibrium of $\mathcal{E}^1$, $p^1 \cdot x_t^1 < p^0 \cdot \xi_t$ for a.e. $t \in \mathcal{I}$.

Set $(\bar{x}, \bar{y}, \bar{p}, \bar{q}) = (x^1, y, p^1, q^1)$, with y_t as in Claim 7.1. Let $K' = \{x \in I\!R^L \mid (0,0) <_{lex} \mathcal{P}x\} \cup \{0\}$. Clearly this is a convex and pointed cone (that is, $-K' \cap K' = \{0\}$). Since for all $t \in \mathcal{J}$, $y_t \in S_{j(t)}(\mathcal{P})$, we have for all $t \in \mathcal{J}$, $Y_{j(t)} - y_t \subset -K'$. For all $t \in \mathcal{J}$, let K_t be the positive hull of $K' \cap (y_t - Y_{j(t)})$. Note that for all $t \in \mathcal{I}$, if $x_t \in P_{i(t)}(\bar{x}_t)$, then $(0,0) <_{lex} \mathcal{P}(x_t - \bar{x}_t)$. For all $t \in \mathcal{I}$, let K_t be the positive hull of $K' \cap (P_{i(t)}(\bar{x}_t) - \bar{x}_t)$. Let $K = \mathrm{cl}\{\mathrm{co} \cup_{t \in \mathcal{I} \cup \mathcal{J}} K_t\}$. Of course K is a convex cone and by the finiteness of the consumption and production sets $K \subset K'$. Thus, $-K \cap K = \{0\}$. For all $t \in \mathcal{I}$, $P_{i(t)}(\bar{x}_t) - \bar{x}_t \subset K$, for all $t \in \mathcal{J}$, $Y_{j(t)} - \bar{y}_t \subset -K$, which ends the proof. $\square$

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