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LONG-LIVED COLLATERALIZED ASSETS AND BUBBLES

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Abstract. When infinite-lived agents trade long-lived assets secured by durable goods, equilibrium exists without any additional debt constraints or uniform impatience conditions on agents' characteristics. Also, price bubbles are absent when physical endowments are uniformly bounded away from zero.

Keywords. Collateralized assets, Existence of equilibrium, Asset pricing bubbles.
JEL Classification. D50, D52.

1. Introduction

Sequential economies with infinite-lived assets have been studied for quite a long time in finance and in macroeconomics. The pioneering models were of two kinds: the overlapping generations models by Samuelson (1958) and Gale (1973) and the infinite-lived agents model by Bewley (1980). The latter inspired a general equilibrium literature that focused on two subtle issues: existence of equilibrium and occurrence of asset price bubbles (see, for instance, Magill and Quinzii (1996), Hernandez and Santos (1996), and Santos and Woodford (1997)).

The previous literature addressed the case of default-free unsecured assets. Generic existence of equilibrium was established under debt-constraints and uniform impatience (Magill and Quinzii (1996) and Hernandez and Santos (1996)). For nicely behaved deflators yielding finite present values of wealth, speculation in assets in positive net supply was ruled out when markets were complete or when agents were uniformly impatient, but bubbles with real effects might occur in the case of assets in zero net supply (see Santos and Woodford (1997) and Magill and Quinzii (1996)).

In this paper we allow for default and consider zero net supply assets whose short sales are collateralized by durable goods, such as mortgages or mortagage related assets (in the spirit of earlier work on collateral by Geanakoplos and Zame (2002) and Araujo, Pascoa and Torres-Martinez (2002)). In this context, the optimization problem of infinite lived agents gains a very nice structure that allows us to approach existence of equilibrium and speculation in a new way. In fact, the

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returns from past actions (namely from the joint operation of collateralizing and short-selling) are always non-negative and, therefore, as in positive dynamic programming, Euler and transversality conditions are not just necessary but also sufficient for individual optimality. That is, a plan that satisfies Euler and transversality conditions is optimal among all budget feasible plans (and not just among those that satisfied also transversality conditions, as was the case in the previous literature when short-sales were allowed but were unsecured by collateral). Moreover, endowments are no longer required to be bounded away from zero, due to the durability of previous endowments.

From the sufficiency of the optimality conditions we establish existence of equilibrium, without imposing debt constraints or uniform impatience requirements. We suppose, nevertheless, that utilities are time and state separable, which has not been assumed in the previous literature on existence of equilibrium (see Magill and Quinzii (1994) or in Hernandez and Santos (1996)). As in the case of short-lived assets (see Araujo, Páscoa and Torres-Martínez (2002) or Kubler and Schmedders (2003)), collateral avoids Ponzi schemes. Note that Ponzi schemes are being ruled out not because the scarcity of collateral goods bounds short sales, but rather because consumers' optimization problems became positive dynamic programming problems.

However, Ponzi schemes were not the only possible reason for non-existence of equilibrium with long-lived assets. In fact, in economies with default-free long-lived assets, where debt constraints requirements and uniform impatience were imposed, equilibrium still failed to exist and only generic existence was guaranteed (see Hernandez and Santos (1996) or Magill and Quinzii (1996)). Two difficulties came up: (i) there were no endogenous upper bounds on short-sales, as the rank of returns matrices became dependent on asset prices; and (ii) finite asset prices might be incompatible with non-arbitrage conditions, as the return matrices of zero-net supply assets could be unbounded along the event-tree (see Hernandez and Santos (1996, Example 3.9)). Collateral avoids also these two additional difficulties, since the scarcity of physical goods assures that collateralized short-sales are bounded (overcoming (i)) and, by non-arbitrage (see below), bounded collateral coefficients end up bounding asset prices (overcoming (ii)).

From the necessity of the optimality conditions we establish the properties that commodity and asset prices should satisfy and find out that asset prices are always bounded by the collateral cost. We use this result and focus on deflators that are compatible with the optimality conditions (and, therefore, known to yield finite present values of wealth). First, we show that mortgages, whose collateral does not have margin calls, are free of price bubbles unless the durable good serving as collateral (or being part of the real payments) has a price bubble itself. Secondly, for more general collateral requirements, speculation is ruled out if endowments are uniformly bounded away from zero.

Finally, note that uniform impatience had played a crucial role in default-free economies when it came to show that debt constraints turned out to be equivalent to imposing the transversality requirements that the optimal plan should verify. That is, under uniform impatience, the chosen default-free plan was optimal among the debt-constrained or transversality-constrained plans that satisfied the budget constraints. In our model, the chosen plan is optimal among all budget feasible plans and we can do without uniform impatience, which is far from being a trivial assumption. Even for separable utility functions and endowments that are uniformly bounded away from zero, the assumption may fail if inter-temporal discounting is not stationary.

The rest of the paper is organized as follows. The next two sections present the model. In Section 4 we discuss a crucial property of the default model: a consumption and portfolio plan is individually optimal if and only if it satisfies Euler inequalities and a transversality condition on its cost. The necessity part is used to characterize asset prices. The sufficiency part, which is the novel result, is used to establish existence of equilibrium without uniform impatience requirements (in Section 5). Our asset pricing characterization (which is analogous to the non-arbitrage valuation studied by Araujo, Fajardo and Páscoa (2005)) is the basis for the definitions of fundamental values and for the results on absence of price bubbles (in Section 6).

2. Infinite Horizon Collateralized Asset Markets

Uncertainty. We consider a discrete time economy with infinite horizon. A date is an element $t \in \{0, 1, \ldots\}$. There is no uncertainty at $t = 0$ and given a history of realization of the states of nature for the first t dates, with $t \geq 1$, $\overline{s}_t = (s_0, \ldots, s_{t-1})$, there is a finite set $S(\overline{s}_t)$ of states of nature that may occur at date t.

A vector $\xi = (t, \overline{s}_t, s)$, where $t \geq 1$ and $s \in S(\overline{s}_t)$, is called a *node* of the economy. There is only one node at $t = 0$, that is denoted by ξ_0. Given $\xi = (t, \overline{s}_t, s)$ and $\mu = (t', \overline{s}_{t'}, s')$, we say that μ is a *successor* of ξ, and write $\mu \geq \xi$, if both $t' \geq t$ and $(\overline{s}_{t'}, s') = (\overline{s}_t, s, \ldots)$. We write $\mu > \xi$ to say that $\mu \geq \xi$ but $\mu \neq \xi$. The set of nodes, called the *event-tree*, is denoted by D.

Let $t(\xi)$ be the date associated with a node $\xi \in D$. Let $\xi^+ := \{\mu \in D : (\mu \geq \xi) \wedge (t(\mu) = t(\xi)+1)\}$. The (unique) predecessor of ξ, with $t(\xi) \geq 1$, is denoted by ξ^- and $D(\xi) = \{\mu \in D : \mu \geq \xi\}$ is the subtree with root ξ. The family of nodes with date T in $D(\xi)$ is denoted by $D_T(\xi)$. Finally, given $T \geq 1$, let $D^T(\xi) := \bigcup_{k=t(\xi)}^{T} D_k(\xi)$, $D^T := D^T(\xi_0)$ and $D_T := D_T(\xi_0)$.

Physical markets. At each node there is a finite ordered set of commodities, L, which can be traded and may suffer transformations at the immediate successors nodes. We allow for goods that are perishable or perfectly durable and also for transformation of some commodities into others.

More formally, for any $\eta \in D$, there is a matrix with non-negative entries $Y_\eta = (Y_\eta(l,l'))_{(l,l')\in L\times L}$ such that, if one unit of good $l \in L$ is consumed at a node ξ, then at each $\mu \in \xi^+$ remain $Y_\mu(l,l)$ units of l and we obtain $Y_\mu(l',l)$ units of each commodity $l' \neq l$. For convenience of notations, given a history of nodes $\{\xi_1,\ldots,\xi_n\}$, with $\xi_{j+1} \in \xi_j^+$, we define Y_{ξ_1,ξ_n} as equal to $Y_{\xi_n}Y_{\xi_{n-1}}\cdots Y_{\xi_2}$, when $n > 1$; and equal to the identity matrix when $n = 1$.

Spot markets for commodity trade are available at each node. Denote by $p_\xi = (p_{\xi,l}\ : l \in L) \in \mathbb{R}^L_+$ the row vector of spot prices at $\xi \in D$ and by $p = (p_\xi : \xi \in D)$ the process of commodity prices.

Financial markets. There is a finite ordered set J of different types of infinite-lived assets. Assets may suffer default but are protected by physical collateral requirements.[1] Assets of a given type have the same promises of real deliveries and the same collateral requirements. Thus, in the absence of default, assets of the same type can be treated as being the same asset. However, when an asset issued at ξ defaults at a successor node $\mu > \xi$, it converts into the respective *collateral*. For this reason, we suppose that, at every node, an asset of each type $j \in J$ can be issued. In this way, we assure that agents can constitute, at any node, new long or short positions on assets of any type. For the sake of simplicity, whenever there is no possible confusion, we will refer to an asset of type j simply as *asset* j.

Assets are in zero net supply. At any $\xi > \xi_0$, real promises associated to one unit of asset $j \in J$ are given by a bundle $A(\xi,j) \in \mathbb{R}^L_+$. Let $(C_{\xi,j};\xi \in D) \in \mathbb{R}^{L\times D}_+$ be the plan of asset' j unitary collateral requirements.

We denote by $q_\xi = (q_{\xi,j},\, j \in J) \in \mathbb{R}^J_+$ the row vector of asset prices at $\xi \in D$, and by $q = (q_\xi,\, \xi \in D)$ a plan of asset prices in the event-tree.

Note that, when assets are short-sold, borrowers have to constitute collateral. In case of default, the depreciated collateral will be seized. Also, others goods delivered by the collateral bundle may also be garnishable. That is, we assume that, in case of default on asset j at node $\xi > \xi_0$, markets seize the *garnishable collateral*, which is given by a bundle $\widehat{C}_{\xi,j}$ that satisfies, $Y_\xi(l,l)C_{\xi^-,j,l} \leq \widehat{C}_{\xi,j,l} \leq Y_\xi(l,\cdot)C_{\xi^-,j}$, $\forall l \in L$. Note that, if Y_ξ is a diagonal matrix (as in Araujo, Páscoa and Torres-Martínez (2002)), then $\widehat{C}_{\xi,j}$ coincides with $Y_\xi C_{\xi^-,j}$. However, when collateral is durable but delivers also perishable commodities at the next nodes, those deliveries might also be or not be seized in case of default. Hence, borrowers will pay and lenders expect to receive the minimum between the value of the garnishable collateral and the market value of the original debt. Thus, the (unitary) nominal

[1]We could have allowed for price dependent collateral requirements and for financial collateral as long as we ruled out self-collateralization (the possibility that an asset ends up of securing itself though a chain of other assets). For more details see Araujo, Páscoa and Torres-Martínez (2005)

payment made by asset j at node $\xi > \xi_0$ is given by $D_{\xi,j}(p,q) := \min\{p_\xi A(\xi,j) + q_{\xi,j}, p_\xi \widehat{C}_{\xi,j}\}$. To shorten notations, let $D_\xi(p,q) := (D_{\xi,j}(p,q), j \in J)$.

Finally, we want to show two simple and important examples of collateral requirements processes contemplated by our framework. First, if for any $\xi \in D$, $C_{\xi,j} = C \in \mathbb{R}^L_+$, then, as collateral guarantees may depreciate along the event-tree, borrowers may need to buy additional physical resources in order to maintain their original short-positions. In some sense, it is similar to the well known market practice of margin calls. Secondly, the case of *mortgage loans*, where $C_{\xi,j} \leq Y_\xi C_{\xi^-,j}$, for any $\xi > \xi_0$. In this case, short-positions can be maintained without need to update the amount of physical guarantees.

Households. There is a finite set, H, of infinite-lived agents that consume commodities and trade assets along the event-tree. Each agent $h \in H$ has a physical endowment processes given by $w^h = (w^h_\xi\,;\, \xi \in D) \in \mathbb{R}^{D\times L}_+$. At each $\xi \in D$, any agent h can choose a plan $z^h_\xi = (x^h_\xi, \theta^h_\xi, \varphi^h_\xi) \geq 0$, where $x^h_\xi := (x^h_{\xi,l}\,;\, l \in L)$ is the *autonomous consumption bundle* (that is, her consumption in excess of required physical collateral) and $(\theta^h_\xi, \varphi^h_\xi) = \Big((\,\theta^h_{\xi,j}\,, \varphi^h_{\xi,j}\,)\,;\, j \in J\Big)$ denotes, respectively, her *long-* and *short-positions* at ξ. Agent h consumption at a node ξ is given by $\widehat{x}^h_\xi = x^h_\xi + \sum_{j\in J} C_{\xi,j}\, \varphi^h_{\xi,j}$.

Given prices (p,q), the objective of consumer h is to maximize her utility function $U^h : \mathbb{R}^{D\times L}_+ \to \mathbb{R}_+ \cup \{+\infty\}$ over the plans $\widehat{x}^h$, by choosing a plan $z^h = (x^h, \theta^h, \varphi^h) \in \mathbb{E} := \mathbb{R}^{D\times L}_+ \times \mathbb{R}^{D\times J}_+ \times \mathbb{R}^{D\times J}_+$ which satisfies the following budget constraints,[2]

$$g^h_{\xi_0}(z^h_{\xi_0}, z^h_{\xi_0^-}; p, q) := p_{\xi_0}(\widehat{x}^h_{\xi_0} - w^h_{\xi_0}) + q_{\xi_0}\left(\theta^h_{\xi_0} - \varphi^h_{\xi_0}\right) \leq 0, \tag{1}$$

and for all $\xi > \xi_0$,

$$g^h_\xi(z^h_\xi, z^h_{\xi^-}; p, q) := p_\xi(\widehat{x}^h_\xi - w^h_\xi - Y_\xi \widehat{x}^h_{\xi^-}) + q_\xi\left(\theta^h_\xi - \varphi^h_\xi\right) - D_\xi(p,q)\left(\theta^h_{\xi^-} - \varphi^h_{\xi^-}\right) \leq 0, \tag{2}$$

where $z^h_{\xi_0^-} := (x^h_{\xi_0^-}, \theta^h_{\xi_0^-}, \varphi^h_{\xi_0^-}) = 0$. The budget set of agent h at prices (p,q), denoted by $B^h(p,q)$, is the collection of plans $(x,\theta,\varphi) \in \mathbb{E}$ such that inequalities (1) and (2) hold. Moreover, without loss of generality, we restrict the price set to $\mathbb{P} := \{(p_\xi, q_\xi)_{\xi\in D} : (p_\xi, q_\xi) \in \Delta^{L+J-1}_+,\ \forall \xi \in D\}$, where Δ^{n-1}_+ denotes the $(n-1)$-dimensional simplex in $\mathbb{R}^n_+$.

Definition 1. *An equilibrium for our economy is given by prices $(p,q) \in \mathbb{P}$ and individual plans $\big((x^h, \theta^h, \varphi^h)\big)_{h\in H} \in \mathbb{E}^H$, such that*
A. For each $h \in H$, $(x^h, \theta^h, \varphi^h) \in Argmax\,\{U^h(\widehat{x}),\, (x,\theta,\varphi) \in B^h(p,q)\}$.

[2]Note that, the non-negativity condition on the autonomous consumption represents the *physical collateral constraint*. In fact, the later requires $\widehat{x}^h_\xi \geq \sum_{j\in J} C_{\xi,j}\, \varphi^h_{\xi,j}$, which is equivalent to $x^h_\xi \geq 0$.

B. Asset markets are cleared. That is, for each $j \in J$,

$$\sum_{h\in H} \left(\theta^h_{\xi,j} - \varphi^h_{\xi,j}\right) \;=\; 0, \quad \forall \xi \geq \xi_0.$$

C. Physical markets are cleared.

$$\sum_{h\in H} \widehat{x}^h_{\xi_0} = \sum_{h\in H} w^h_{\xi_0};$$

and, at each $\xi \neq \xi_0$,

$$\sum_{h\in H} \widehat{x}^h_{\xi} = \sum_{h\in H} \left(w^h_{\xi} + Y_\xi \widehat{x}^h_{\xi^-}\right).$$

3. Assumptions on Agents' Characteristics

As commodities can be durable goods, the traditional assumption that individual endowments of commodities are interior points can be replaced by the weaker assumption that requires only individual accumulated resources to be interior points. Moreover, to assure the existence of equilibrium, we do not need to impose any uniform lower bound in the aggregate cumulated resources. Thus we allow for durable commodities whose aggregate resources converge to zero.

Assumption A. *For each $(h,\xi) \in H \times D$, given the history of realization of states of nature up to node ξ, $F_\xi := \{\xi_0, \ldots, \xi^-, \xi\}$, we have that $W^h_\xi := \sum_{\mu\in F_\xi} Y_{\mu,\xi}\, w^h_\mu \gg 0$. Moreover, for each $(\xi, j) \in D \times J$, $C_{\xi,j} \neq 0$.*

The aggregated resources up to a node ξ need to take into account the streams of real resources generated by the financial endowments. Thus, an *upper bound* for the bundle of aggregate physical resources up to a node ξ is given by $\mathbb{W}_\xi := \sum_{h\in H} W^h_\xi + \sum_{\mu\in F_\xi} Y_{\mu,\xi} \sum_{j\in J} b^j_\mu e_j$, where $b^j_{\xi_0} = 0$ and $b^j_\xi = (b^j_{\xi,l})_{l\in L}$, with $b^j_{\xi,l} = \max\left\{\widehat{C}_{\xi,j,l};\ A(\xi,j)_l\right\}$, for each $\xi > \xi_0$.

Assumption B. *The utility function of each $h \in H$ is separable in time and in states of nature, in the sense that $U^h(\widehat{x}) := \sum_{\xi\in D} u^h_\xi(\widehat{x}(\xi))$, where functions $u^h_\xi : \mathbb{R}^L_+ \to \mathbb{R}_+$ are strictly concave, continuous, and strictly increasing. Also, $\sum_{\xi\in D} u^h_\xi(\mathbb{W}_\xi) < +\infty$.*

Under hypotheses above, uniform impatience conditions imposed by Hernandez and Santos (1996, Assumption C.3), Magill and Quinzzi (1996, Assumptions B2 and B4) and Santos and Woodford (1997, Assumption A.2) do not necessarily hold.[3] For example, given any $u : \mathbb{R}^L_+ \to \mathbb{R}_+$ strictly concave, continuous, and strictly increasing, consider the function $U(\widehat{x}) := \sum_{\xi\in D} \beta_{t(\xi)}\rho(\xi)u(\widehat{x}(\xi))$,

[3] For instance, using the notation of Assumption B, in a context where aggregated physical endowments were exogenously fixed and given by the plan $(W_\xi)_{\xi\in D}$, Hernandez and Santos (1996) imposed the following assumption of uniform impatience: There exists $\sigma \times K \in [0,1) \times \mathbb{R}_{++}$ such that, for any plan of consumption $(\widehat{x}_\xi)_{\xi\in D}$ for which

where $(\beta_t)_{t\geq 0} \in \mathbb{R}^{\mathbb{N}}_{++}$, $\rho(\xi_0) = 1$ and, for each $\xi \in D$, $\rho(\xi) = \sum_{\mu\in\xi^+} \rho(\mu)$. Then, when physical resources are uniformly bounded along the event-tree and $\sum_{t\geq 0} \beta_t$ is finite, Assumption B holds. If in addition individual endowments are uniformly bounded away from zero, Assumption A is satisfied. However, in this context, the function U may fail to satisfy uniform impatience condition when inter-temporal discount factors are not stationary. Santos and Woodford (1997, example 4.5) gave an example that illustrates this possibility.

4. Individual Optimality

In this section we present necessary and sufficient conditions for individual optimality. As in positive dynamic programming theory, we will show that the default structure gives inter-temporal Lagrangian functions a sign property under which Euler inequalities jointly with a transversality condition are not just necessary but also sufficient to guarantee the optimality of a consumption-portfolio plan.

Let $\mathbb{Z} := \mathbb{R}^L \times \mathbb{R}^J \times \mathbb{R}^J$. Given prices $(p, q) \in \mathbb{P}$, it follows from the arguments of the previous section that the objective of the agent h is to find a plan $(z^h_\xi)_{\xi\in D} \in \mathbb{Z}^D$ in order to solve

$$P^h_{(p,q)} \qquad \begin{array}{ll} \max & \sum_{\xi\in D} v^h_\xi(z_\xi) \\ \text{s.t.} & \left\{ \begin{array}{lll} g^h_\xi\left(z_\xi, z_{\xi^-}; p, q\right) & \leq & 0, \quad \forall \xi \in D, \\ z_\xi = (x_\xi, \theta_\xi, \varphi_\xi) & \geq & 0, \quad \forall \xi \in D, \;\; z_{\xi_0^-} = 0. \end{array} \right. \end{array}$$

where $v^h_\xi : \mathbb{Z} \to \mathbb{R} \cup \{-\infty\}$ is defined at any $z_\xi = (x_\xi, \theta_\xi, \varphi_\xi) \in \mathbb{Z}$ by

$$v^h_\xi(z_\xi) = \left\{ \begin{array}{ll} u^h_\xi\left(x_\xi + \sum_{j\in J} C_{\xi,j}\,\varphi_{\xi,j}\right) & \text{if}\, x_\xi + \sum_{j\in J} C_{\xi,j}\,\varphi_{\xi,j} \geq 0 \\ -\infty & \text{in other case.} \end{array} \right.$$

For each real number $\gamma \geq 0$, let $\mathcal{L}^h_\xi(\cdot, \gamma; p, q) : \mathbb{Z} \times \mathbb{Z} \to \mathbb{R}$ be the *Lagrangian function* associated to consumer problem at node ξ, which is defined by

$$\mathcal{L}^h_\xi(z_\xi, z_{\xi^-}, \gamma; p, q) = v^h_\xi(z_\xi) - \gamma\, g^h_\xi(z_\xi, z_{\xi^-}; p, q). \tag{3}$$

Since under Assumption B the function $\mathcal{L}^h_\xi(\cdot, \gamma; p, q)$ is concave, we can consider its super-differential set at any point $(z_\xi, z_{\xi^-}) \in \mathbb{Z} \times \mathbb{Z}$, $\partial\mathcal{L}^h_\xi(z_\xi, z_{\xi^-}, \gamma; p, q)$, which is defined as the set of vectors $(\mathcal{L}'_{\xi,1}, \mathcal{L}'_{\xi,2}) \in \mathbb{Z} \times \mathbb{Z}$ such that, for all pair $(z'_\xi, z'_{\xi^-}) \in \mathbb{Z} \times \mathbb{Z}$,

$$\mathcal{L}^h_\xi(z'_\xi, z'_{\xi^-}, \gamma; p, q) - \mathcal{L}^h_\xi(z_\xi, z_{\xi^-}, \gamma; p, q) \leq (\mathcal{L}'_{\xi,1}, \mathcal{L}'_{\xi,2}) \cdot \left((z'_\xi, z'_{\xi^-}) - (z_\xi, z_{\xi^-})\right). \tag{4}$$

$\widehat{x}_\xi \leq W_\xi$, $\forall \xi \in D$, we have that

$$u^h_\xi(\widehat{x}_\xi + KW_\xi) + \sum_{\mu>\xi} u^h_\mu(\sigma\,\widehat{x}_\mu) > \sum_{\mu\geq\xi} u^h_\mu(\widehat{x}_\mu), \;\; \forall h \in H.$$

Essentially, the above vectors $\mathcal{L}'_{\xi,1}$ and $\mathcal{L}'_{\xi,2}$ are partial super-gradients with respect to the current and past decision variables, respectively.

DEFINITION 2. *Given $(p,q)\in\mathbb{P}$, $(\gamma^h_\xi)_{\xi\in D}\in\mathbb{R}^D_{++}$ is a plan of Kuhn-Tucker multipliers associated with $(z^h_\xi)_{\xi\in D}\in\mathbb{Z}^D$ if there is $(\mathcal{L}'_{\xi,1},\mathcal{L}'_{\xi,2})_{\xi\in D}\in\prod_{\xi\in D}\partial\mathcal{L}^h_\xi(z^h_\xi,z^h_{\xi^-},\gamma^h_\xi;p,q)$ such that, for any $\xi\in D$, $\gamma^h_\xi g^h_\xi(z^h_\xi,\, z^h_{\xi^-};p,q)=0$ and the following transversality and Euler conditions hold,*

(TC)
$$\lim_{T\to+\infty}\sum_{\mu\in D_T(\xi_0)}\mathcal{L}'_{\mu,1}\,z^h_\mu=0.$$

(EE)
$$\mathcal{L}'_{\xi,1}+\sum_{\mu\in\xi^+}\mathcal{L}'_{\mu,2}\le 0,\qquad \left(\mathcal{L}'_{\xi,1}+\sum_{\mu\in\xi^+}\mathcal{L}'_{\mu,2}\right)z^h_\xi=0,\qquad \forall\xi\in D.$$

PROPOSITION 1. *Suppose that Assumptions A and B hold. Given $(p,q)\in\mathbb{P}$, take a plan $(z^h_\xi)_{\xi\in D}=(x^h_\xi,\theta^h_\xi,\,\varphi^h_\xi)_{\xi\in D}\in B^h(p,q)$.*

(i) *If $(z^h_\xi)_{\xi\in D}$ gives a finite optimum to $P^h_{(p,q)}$, then there is a plan of Kuhn-Tucker multipliers associated with $(z^h_\xi)_{\xi\in D}$.*

(ii) *Reciprocally, the plan $(z^h_\xi)_{\xi\in D}$ solves $P^h_{(p,q)}$ when there are Kuhn-Tucker multipliers associated with it. Also, if $\widehat{x}^h_\xi\le\mathbb{W}_\xi$, for each $\xi\in D$, then the optimum value is finite.*

(iii) *Given Kuhn-Tucker multipliers , $(\gamma^h_\xi)_{\xi\in D}$, associated with $(z^h_\xi)_{\xi\in D}$, $\sum_{\xi\in D}\gamma^h_\xi p_\xi w^h_\xi<\infty$.*

The proof that existence of Kuhn-Tucker multipliers implies individual optimality depends crucially on the following *sign property* of Lagrangian functions, which holds at any $\xi\in D$: Given prices $(p,q)\in\mathbb{P}$ and a plan $(z_\xi)_{\xi\in D}\in\mathbb{Z}^D$,

$$\forall\gamma\in\mathbb{R}_{++}\,\forall\xi>\xi_0:\quad(\mathcal{L}'_{\xi,1},\mathcal{L}'_{\xi,2})\in\partial\mathcal{L}^h_\xi(z_\xi,z_{\xi^-},\gamma;p,q)\Longrightarrow\mathcal{L}'_{\xi,2}\ge 0.$$

This property is very specific to our model. In fact, as for each $j\in J$ effective returns $D_{\xi,j}(p,q)$ are not greater than the respective garnishable collateral values, the joint returns from actions taken at immediately preceding nodes are non-negative (for more details, see Appendix A).

Condition (TC) is not a constraint that is imposed together with the budget restrictions (as was the case in Hernandez and Santos (1996) or Magill and Quinzii (1996)), it is rather a property that optimal plans should satisfy. Moreover, as the deflated value of endowments is summable (item (iii) of Proposition 1) condition (TC) can be rewritten as requiring that, as time tends to infinity, the deflated cost of the autonomous consumption goes to zero,

(TC_x)
$$\lim_{T\to+\infty}\sum_{\xi\in D_T(\xi_0)}\gamma^h_\xi p_\xi\,x^h_\xi\quad=\quad 0;$$

jointly with the cost of the joint operation of constituting collateral and short-selling,

$$(\mathrm{TC}_\varphi) \qquad \lim_{T\to+\infty} \sum_{\xi\in D_T(\xi_0)} \gamma_\xi^h \left(p_\xi \sum_{j\in J} C_{\xi,j}\varphi_{\xi,j}^h - q_\xi \varphi_\xi^h \right) = 0;$$

and the cost of asset purchases,

$$(\mathrm{TC}_\theta) \qquad \lim_{T\to+\infty} \sum_{\xi\in D_T(\xi_0)} \gamma_\xi^h q_\xi \, \theta_\xi^h = 0,$$

where $z_\xi^h = (x_\xi^h, \theta_\xi^h, \varphi_\xi^h)$ (see Appendix A).

We end this section with a characterization of commodity and asset prices.

Corollary 1. (Asset pricing conditions) *Suppose that Assumptions A and B hold. Fix prices $(p,q)\in\mathbb{P}$ such that, for some $h\in H$, $P^h_{(p,q)}$ has a finite optimum. Then, there exist, for any $\xi\in D$, strictly positive deflators γ_ξ and non-pecuniary returns $\alpha_\xi = (\alpha_{\xi,l})_{l\in L}\in\mathbb{R}^L_{++}$ such that, for each $(l,j)\in L\times J$,*

$$(5) \qquad \gamma_\xi p_{\xi,l} \geq \sum_{\mu\in\xi^+} \gamma_\mu p_\mu Y_\mu(\cdot,l) + \alpha_{\xi,l};$$

$$(6) \qquad \gamma_\xi q_{\xi,j} \geq \sum_{\mu\in\xi^+} \gamma_\mu D_{\mu,j}(p,q);$$

$$(7) \qquad \gamma_\xi (p_\xi C_{\xi,j} - q_{\xi,j}) \geq \sum_{\mu\in\xi^+} \gamma_\mu \left(p_\mu Y_\mu C_{\xi,j} - D_{\mu,j}(p,q)\right) + \alpha_\xi C_{\xi,j}.$$

Moreover, for any $(\xi,j)\in D\times J$, conditions (6) or (7) are strict inequalities only when inequality (5) is strict for some $l\in L$ for which $C_{\xi,j,l}>0$.

This result is a direct consequence of the existence of Kuhn-Tucker multipliers associated with agent h optimal problem. Indeed, as we prove in Appendix A, conditions (5)-(7) are essentially equal to the Euler conditions. Clearly, there may exist deflators $(\gamma_\xi)_{\xi\in D}$ satisfying (5)-(7) that are not compatible with the transversality condition (TC) and, therefore, do not coincide with a plan of Kuhn-Tucker multipliers. In fact, that broader set of deflators satisfying (5), (6) and (7), can be obtained by a non-arbitrage argument, as in the two dates model by Araujo, Fajardo and Páscoa (2005). However, if we pick agent h Kuhn-Tucker multipliers, it follows that non-linearities on asset prices can only arise as a consequence of binding collateral constraints (or, in other words, binding sign constraints on the autonomous consumption, determining positive shadow prices that are responsible for the strict inequality in (5)).

Under monotonicity of preferences, inequalities (6) and (7) are financial non-arbitrage conditions. Thus, by analogy to Magill and Quinzzi (1996) or Santos and Woodford (1997), for some readers it might seem natural to use these two conditions only to analyze the existence of rational asset pricing bubbles. However, since in our model assets are real and commodities may be infinitely durable, we need to understand the asymptotic behavior of commodity prices. To do this, we must also consider inequality (5). Note that in this condition the non-pecuniary returns, $(\alpha_{\xi,l})_{l\in L}$, are not vague concepts and can actually be related to marginal utility gains of some agent (by Proposition 1 (i)).

Definition 3. A plan $\Gamma := (\gamma_\xi)_{\xi\in D} \in \mathbb{R}^D_{++}$ is a *process of valuation coefficients* at prices $(p,q) \in \mathbb{P}$ if there is, for each $\xi \in D$, a vector $(\alpha_{\xi,l})_{l\in L} \in \mathbb{R}^L_{++}$ such that inequalities (5)-(7) hold.

Thus, any plan of Kuhn-Tucker multipliers of an agent h, denoted by $\Gamma^h = (\gamma^h_\xi)_{\xi\in D}$, is a process of valuation coefficients.

For convenience of future notations, given any process Γ of valuation coefficients, for each $\xi \in D$, let $\eta(\Gamma,\xi) = (\eta_x(\Gamma,\xi,l);\, \eta_\theta(\Gamma,\xi,j);\, \eta_\varphi(\Gamma,\xi,j))_{(l,j)\in L\times J}$ be the vector defined by

$$\begin{aligned}
\eta_x(\Gamma,\xi,l) &= \gamma_\xi p_{\xi,l} - \sum_{\mu\in\xi^+} \gamma_\mu p_\mu Y_\mu(\cdot,l) - \alpha_{\xi,l};\\
\eta_\theta(\Gamma,\xi,j) &= \gamma_\xi q_{\xi,j} - \sum_{\mu\in\xi^+} \gamma_\mu D_{\mu,j}(p,q);\\
\eta_\varphi(\Gamma,\xi,j) &= \gamma_\xi (p_\xi C_{\xi,j} - q_{\xi,j}) - \sum_{\mu\in\xi^+} \gamma_\mu \left(p_\mu Y_\mu C_{\xi,j} - D_{\mu,j}(p,q)\right) - \alpha_\xi C_{\xi,j}.
\end{aligned}$$

Note that, when $\Gamma = \Gamma^h$, for some agent $h \in H$, $\eta(\Gamma^h,\xi)$ is the vector of shadow prices associated with the collateral constraints and the sign constraints on long and short positions, respectively. The shadow prices $\eta_\theta(\Gamma^h,\xi,j)$ of the sign constraint on long positions are actually the shadow prices of the restriction preventing unsecured short-sales (and are equal to $\eta_x(\Gamma^h,\xi,l)C_{\xi,j} - \eta_\varphi(\Gamma^h,\xi,j)$).

Finally, it is important to remark that equation (7) and Assumption B imply that, at each $\xi \in D$,

$$p_\xi C_{\xi,j} > q_{\xi,j}, \quad \forall j \in J. \tag{8}$$

Thus, the collateral cost must exceed the asset price. This condition will be crucial in relating the occurrence of asset price bubbles to the asymptotic behavior of commodity prices.

5. Equilibrium Existence

As we point out earlier, when assets live more than one period and agents are infinite lived, three difficulties came up in the literature on equilibrium in default-free economies that made the authors assert only the generic existence of equilibrium in economies where agents are uniformly impatient and for debt-constrained (or transversality constrained) portfolio plans (as in Hernandez and Santos (1996) and Magill and Quinzii (1996)).[4] First, when assets live several periods, the rank of the returns matrix will depend on asset prices and, therefore, unless short-sales are bounded, equilibrium existed, in the default-free model, only for a generic set of economies. Second, Ponzi schemes could occur, if either debt (or transversality) restrictions were not imposed or agents were not uniform impatient. Third, as Hernandez and Santos (1996) pointed out, when asset return matrices are not bounded along the event-tree, equilibria might not exist when infinite-lived real assets are in zero net supply.[5]

However, when assets are collateralized, these difficulties are avoided.

Theorem 1. *Under Assumptions A and B there exists an equilibrium.*

Note that, even in the case of single period assets (see Geanakoplos and Zame (2002)), collateral circumvented the problems associated to the price-dependence of the rank of the return matrices. In fact, collateral is scarce in equilibrium and, therefore, we will have a natural (endogenous) short-sales constraint. Moreover, collateral rules out Ponzi schemes, as it did in the case of single-period assets (see Araujo, Páscoa and Torres-Martínez (2002)). Finally, the existence of collateral guarantees dispenses with any uniform bounds on assets' promised returns, as the asset price is bounded by the discounted value of the depreciated collateral at the next date, plus perhaps some shadow price of the collateral constraint.

6. Speculative Bubbles in Prices

As in Magill and Quinzii (1996) and Santos and Woodford (1997), speculation is defined as a deviation of the equilibrium price from the *fundamental value* of the asset, which is the deflated value of future payments and services that the asset yields. We define fundamental values as a function of the chosen vector of valuation coefficients. Differently from Santos and Woodford (1997), we do not focus on non-personalized non-arbitrage kernel deflators (which do not take into account the possibility of frictions arising from binding debt constraints). Instead, we look at the personalized deflator induced by the Kuhn-Tucker multipliers, which may be a non-arbitrage kernel deflator and may be the unique such kernel deflator (in the absence of frictions and when markets are complete). Our more general results (under incomplete markets, when personalized deflators are different), address the occurrence of bubbles for specific personalized Kuhn-Tucker deflators.

[4]Hernandez and Santos (1996) were also able to show the existence of equilibrium in the special case where the asset structure consists of a single infinite lived real asset in positive net supply.

[5]In fact, the asset price can be shown to be the series of discounted real returns and would be unbounded, unless marginal rates of substitution tend to zero quickly enough (which would be the case if the asset's net supply were positive, inducing unbounded additional resources).

To simplify our analysis, we suppose that, if a commodity consumed at ξ is transforming itself into other goods at the immediate successors nodes of ξ, then these goods are one-period perishable.

Assumption C. *Given $(\mu,l)\in(D\setminus\{\xi_0\})\times L$, if there is $l'\neq l$ such that $Y_\mu(l',l)\neq 0$, then at μ the commodity l' is one-period perishable, in the sense that $Y_\eta(\cdot,l')=0$, $\forall\eta\in\mu^+$.*

Essentially, this restriction guarantees that fundamental values of commodities may be easily defined in terms of future payments and rental services. Otherwise, the value of payments generated by a good may include speculative terms, induced by the transformation of the good into a durable commodity that has a price bubble. Clearly, the fundamental value of a durable good could be defined in the two extreme cases, when it does not transform itself into other commodities or when it is allowed to generate other durable goods. We model here is an intermediate situation and have in mind situations such as a farm that produces agricultural goods or a building used by commercial or industrial firms producing perishable goods.

Speculation in durable goods. The fundamental value at $\xi\in D$ of any commodity $l\in L$ takes into account both the frictions that will be generated in $D(\xi)$ jointly with the payments that will be delivered when l is transformed into another goods.

More formally, given prices $(p,q)\in\mathbb{P}$ and a process of valuation coefficients Γ, the rental services that one unit of commodity l generates at a node $\mu\in D(\xi)$ are given by

$$\left(p_{\mu,l}-\sum_{\nu\in\mu^+}\frac{\gamma_\nu}{\gamma_\mu}\,p_\nu Y_\nu(\cdot,l)\right)=\eta_x(\Gamma,\mu,l)+\alpha_{\mu,l}.$$

On the other hand, the payments that an agent that holds one unit of commodity l at μ^- receives at node $\mu>\xi$ are given by $\sum_{l'\neq l}p_{\mu,l'}Y_\mu(l',l)$. Moreover, one unit of $l\in L$ at $\xi\in D$ is transforming into $a_l(\xi,\mu)$ units of the same commodity at a node $\mu\in D(\xi)$, where

$$a_l(\xi,\mu)=\begin{cases}\prod_{\xi<\eta\leq\mu}Y_\eta(l,l) & \text{if}\,\mu>\xi,\\ 1 & \text{otherwise.}\end{cases}$$

Therefore, under Assumption C and for a process of valuation coefficients Γ, the *fundamental value of commodity l at node ξ* is defined by

$$F_l(\xi,p,q,\Gamma)\;=\;\sum_{\mu\in D(\xi)}\frac{\gamma_\mu}{\gamma_\xi}\left(\eta_x(\Gamma,\mu,l)+\alpha_{\mu,l}\right)\,a_l(\xi,\mu)+\sum_{\mu>\xi}\frac{\gamma_\mu}{\gamma_\xi}\sum_{l'\neq l}p_{\mu,l'}Y_\mu(l',l)a_l(\xi,\mu^-).$$

Furthermore, for each $T>t(\xi)$,

$$p_{\xi,l}=\sum_{\mu\in D^T(\xi)}\frac{\gamma_\mu}{\gamma_\xi}\left(\eta_x(\Gamma,\mu,l)+\alpha_{\mu,l}\right)\,a_l(\xi,\mu)+\sum_{\mu\in D^T(\xi)\setminus\{\xi\}}\frac{\gamma_\mu}{\gamma_\xi}\sum_{l'\neq l}p_{\mu,l'}Y_\mu(l',l)a_l(\xi,\mu^-)$$

$$+ \sum_{\mu \in D_{T+1}(\xi)} \frac{\gamma_\mu}{\gamma_\xi} p_\mu Y_\mu(\cdot,\, l) a_l(\xi, \mu^-).$$

Since, independently of T, the last term on the right hand side of the equation above is non-negative, it follows that, for any choice of Γ the fundamental value of commodity l is well defined and less than or equal to the unitary price. Also, taking the limit as T goes to infinity, we conclude that, $p_{\xi,l} = F_l(\xi, p, q, \Gamma) + \lim_{T \to +\infty} \sum_{\mu \in D_T(\xi)} \frac{\gamma_\mu}{\gamma_\xi} p_\mu Y_\mu(\cdot,\, l) a_l(\xi, \mu^-)$.

Definition 3. *Given a process Γ of valuation coefficients, we say that the price of commodity $l \in L$ has a Γ-bubble at node ξ when $p_{\xi,l} > F_l(\xi, p, q, \Gamma)$.*

Characterization of bubbles on commodity prices.

There is a Γ-bubble on commodity $l \in L$ price at node $\xi \in D$ if and only if

$$\lim_{T \to +\infty} \sum_{\mu \in D_T(\xi)} \frac{\gamma_\mu}{\gamma_\xi} p_\mu Y_\mu(\cdot,\, l) a_l(\xi, \mu^-) > 0.$$

A commodity l has *finite durability* at a node ξ if there exists $N > 0$ such that $a_l(\xi, \mu) = 0$ for all $\mu \in D(\xi) \setminus D^N(\xi)$. It follows from the characterization above that, under Assumption C, commodities with finite durability at ξ are free of bubbles.[6] For commodities with infinite durability, sufficient conditions for the absence of bubbles are given by the next result.

Theorem 2. *Given equilibrium prices $(p, q) \in \mathbb{P}$, suppose that Assumptions A, B and C hold. A sufficient condition for commodities to be free of Γ-bubbles in $D(\xi)$ is that,*

$$\exists h \in H, \qquad \sum_{\mu \in D(\xi)} \frac{\gamma_\mu}{\gamma_\xi} p_\mu W_\mu^h < +\infty. \tag{9}$$

Given $h \in H$, commodities are free of Γ^h-bubbles in $D(\xi)$ if any of the following conditions hold,

(i) *At any node, agent h receives at least a fraction $k \in (0, 1)$ of aggregated endowments. That is, $\kappa W_\mu^h \leq w_\mu^h$ for all $\mu \in D(\xi)$.*

(ii) *Cumulated depreciation factors $Y_{\xi,\mu}$ are uniformly bounded by above in $D(\xi)$ and new endowments, $(w_\mu^h)_{\mu \in D(\xi)}$, are uniformly bounded away from zero in $D(\xi)$.*

(iii) *Individuals' inter-temporal marginal rates of substitution coincide along the event-tree, i.e., given $h' \in H$ there is $\pi > 0$ such that, $(\pi \gamma_\xi^h)_{\xi \in D}$ is a plan of Kuhn-Tucker multipliers for h'.*

[6] When Assumption C is not satisfied, even commodities with finite durability may have bubbles, as may transform into other goods with infinite durability whose prices have bubbles.

PROOF. Fix $\eta \geq \xi$ and $l \in L$. Assume that condition (9) holds. It follows by Assumption A that, for each $T > t(\eta)$,

$$\sum_{\mu \in D_T(\eta)} \frac{\gamma_\mu}{\gamma_\eta} p_\mu Y_\mu(\cdot, l) a_l(\eta, \mu^-) \leq \frac{1}{W^h_{\eta,l}} \sum_{\mu \in D_T(\eta)} \frac{\gamma_\mu}{\gamma_\eta} p_\mu W^h_\mu.$$

Taking the limit as T goes to infinity, we conclude that $p_{\eta,l}$ is free of Γ-bubbles.

Given $h \in H$, if (i) holds, it follows from item (iii) of Proposition 1 that condition (9) is satisfied, that concludes the proof. Also, if (ii) is satisfied, item (iii) of Proposition 1 assure that bubbles are ruled out. Finally, suppose that equilibrium individual marginal rates of substitution coincide along the event-tree. Then, transversality conditions (TC_x), (TC_φ) and (TC_θ) hold, for all agents under a *same* deflator. Adding up these three conditions across all agents, we get condition (9) above. □

Condition (iii) in the above theorem requires the processes of individuals' Kuhn-Tucker multipliers to be collinear. It is well known that in unrestricted financial markets, this condition is a characteristic property of complete markets and, therefore, equivalent to the property that the rank of the matrix of returns of non-redundant assets should be equal to the number of immediate successor nodes. However, in the presence of binding financial constraints this equivalence may no longer hold. Giménez (2003) made this point in the context of short-sales constraints and gave examples of equilibrium where the above returns matrix had full rank but the presence of a shadow price for these constraints led to multiplicity of multipliers for each agent and non-collinear multipliers across agents. The markets illustrated in those examples were referred by Giménez (2003) as *technically incomplete*, along the lines of an earlier discussion done by Santos and Woodford (1992, 1996). In our context, the collateral constraint might be binding as well and if the respective shadow price were non-zero, the uniqueness of the Kuhn-Tucker multipliers process would no longer be guaranteed by a full-rank property of the returns matrix. Hence, our condition (iii) requires more than just that full-rank property, it requires completeness in the stricter sense proposed by Giménez (2003) for asset-constrained economies.

Finally, it should be noticed that the durable goods are positive net supply assets but other positive net supply assets, say securities like stocks or bonds, could have been used instead to serve as collateral. To preserve the positivity features of the dynamic programming problem of consumers, short sales of those securities should be prevented. Existence of equilibrium follows and, in this case, the fundamental value of a security serving as collateral would be equal to the series of deflated security dividends plus the series of the shadow prices of the collateral constraints. To be more precise, a long-lived security with prices $(\pi_\xi; \xi \in D)$ and nominal dividends $(B_\xi; \xi \in D \setminus \{\xi_0\})$ (which may depend on prices and could be the market value of a real promise) would be added, prevented from being shorted but serving to secure short sales of the promises according to the constraints

$\phi_\xi \geq \sum_{j\in J} K_{\xi,j}\varphi_{\xi,j}$, $\forall \xi \in D$, where ϕ_ξ stands for the security position at node ξ and $K_{\xi,j}$ is the collateral coefficient relative to the promise $j \in J$. Let δ_μ be the shadow price of this constraint at node $\mu \in D$. Then, under a process of valuation coefficients $\Gamma = (\gamma(\mu); \mu \in D)$, the fundamental value of the security at a node $\xi \in D$ became $\sum_{\mu>\xi} \frac{\gamma(\mu)}{\gamma(\xi)} B_\mu + \frac{1}{\gamma(\xi)} \sum_{\mu\geq\xi} \delta_\mu$.That is, the role of the security as collateral distorts its fundamental value, giving it a value above the present value of payoffs. In particular, if the collateral were fiat money, its fundamental value would be the series of shadow prices of the collateral constraints (as was previously pointed out by Iraola (2008) and Araujo, Páscoa and Torres-Martínez (2005)).

Asset Pricing Bubbles. The fundamental value of an asset is the present value of its future yields and shadow prices of the restriction preventing unsecured short sales (a measure of the willingness to go short without constituting collateral). Future yields are the perishable goods directly or indirectly delivered by the asset. Real payments of perishable commodities are the yields directly delivered. Indirect delivered yields are the perishable commodities obtained by the transformations of real payments into other goods, or by the transformation of these goods into others and so on. These goods are received as an original promise or as a collateral garnishment, and are unambiguously anticipated except in the *borderline case*, when the value of the promise equals the garnishable collateral value. Thus, the fundamental value would depend not just on the process of valuation coefficients but also on the believed *delivery rates* for the borderline nodes. However, in the borderline case, each agent does not care about this choice and does not know what are the other agents' choices. Thus, we assume, for simplicity, that in borderline case all borrowers pay their promises and, therefore, the associated deliveries of durable goods are given by the original promises.

Given equilibrium prices (p, q) define a price dependent process $\tau(p, q) = (\tau_{\xi,j}(p, q)) \in [0, 1]^{(D\setminus\{\xi_0\})\times J}$ as

$$\tau_{\xi,j}(p,q) = \begin{cases} 1 & \text{if } p_\xi A(\xi,j) + q_{\xi,j} \leq p_\xi \widehat{C}_{\xi,j}\,, \\ 0 & \text{if } p_\xi A(\xi,j) + q_{\xi,j} > p_\xi \widehat{C}_{\xi,j}. \end{cases}$$

Under prices (p, q), the physical bundle that one unit of asset j negotiated at node ξ delivers at $\mu \in \xi^+$, namely $PD_{\mu,j}(p, q)$, consists of the part of the promises $A_{\mu,j}$ that are effectively delivered and also of the physical deliveries made by the garnished collateral. More precisely, $PD_{\mu,j}(p, q) = \tau_{\mu,j}(p, q)A_{\mu,j} + (1 - \tau_{\mu,j}(p, q))\widehat{C}_{\xi,j}$.

We have that, $D_{\mu,j}(p, q) = p_\mu PD_{\mu,j}(p, q) + \tau_{\mu,j}(p, q)\, q_{\mu,j}$. Using inequality (6) we obtain that,

$$\begin{aligned} q_{\xi,j} &= \sum_{\mu\in\xi^+} \frac{\gamma_\mu}{\gamma_\xi} \left(p_\mu PD_{\mu,j}(p,q) + \tau_{\mu,j}(p,q) q_{\mu,j}\right) + \frac{\eta_\theta(\Gamma,\xi,j)}{\gamma_\xi} \\ &= \sum_{\mu>\xi} \left(\prod_{\xi<\eta<\mu} \tau_{\eta,j}(p,q) \right) \left(\frac{\gamma_\mu}{\gamma_\xi} p_\mu PD_{\mu,j}(p,q) + \frac{\eta_\theta(\Gamma,\mu^-,j)}{\gamma_\xi} \right) \end{aligned}$$

$$+\lim_{T\to+\infty}\sum_{\mu\in D_T(\xi)}\frac{\gamma_\mu}{\gamma_\xi}\,q_{\mu,j}\prod_{\xi<\eta\le\mu}\tau_{\eta,j}(p,q)\,.$$

Under Γ, the fundamental value of an asset $j\in J$ at a node ξ, $F_j(\xi,p,q,\Gamma)$, is defined by

$$F_j(\xi,p,q,\Gamma)=\sum_{\mu>\xi}\left(\prod_{\xi<\eta<\mu}\tau_{\eta,j}(p,q)\right)\left[\frac{\gamma_\mu}{\gamma_\xi}\sum_{l\in L}F_l(\mu,p,q,\Gamma)\,PD_{\mu,j}(p,q)_l+\frac{\eta_\theta(\mu^-,j)}{\gamma_\xi}\right]. \tag{10}$$

It follows that, independently of Γ, the fundamental value at ξ is always well defined and less than or equal to the unitary asset price, $q_{\xi,j}$.

Definition 4. *Given equilibrium prices $(p,q)\in\mathbb{P}$, we say that the price of asset $j\in J$ has a Γ-bubble at a node ξ when $q_{\xi,j}>F_j(\xi,p,q,\Gamma)$.*

By definition, a bubble on asset j may be a consequence of a bubble in a commodity—used as collateral or that is part of the real promises— or may be generated by asymptotically positive asset prices. As assets are backed by physical collateral, the non-arbitrage condition given by equation (8) allows us to find a relationship between the asymptotic value of asset prices and the asymptotic value of collateral bundles. For this reason, and differently from what happens in models without default, the existence of bubbles in financial markets is strongly related to the existence of speculation in physical markets.

Theorem 3. *Given equilibrium prices $(p,q)\in\mathbb{P}$, suppose that Assumptions A, B and C hold. The price of asset $j\in J$ is free of Γ-bubbles in $D(\xi)$ if the following conditions hold,*

$$\exists h\in H:\quad\sum_{\mu\in D(\xi)}\frac{\gamma_\mu}{\gamma_\xi}\,p_\mu\,W^h_\mu<+\infty,\qquad \textit{and}\qquad \lim_{T\to+\infty}\sum_{\mu\in D_T(\xi)}\frac{\gamma_\mu}{\gamma_\xi}\,q_{\mu,j}=0.$$

Given $h\in H$, asset j's price is free of Γ^h-bubbles in $D(\xi)$ if any of the following conditions hold,

(i) *Commodities are free of Γ-bubbles in $D(\xi)$ and asset j is a mortgage loan in $D(\xi)$, i.e., $C_{\mu,j}\le Y_{\xi,\mu}C_{\xi,j}$, for any $(\mu,l)\in(D(\xi)\setminus\{\xi\})\times L$.*

(ii) *Collateral requirements $(C_{\mu,j})_{\mu\in D(\xi)}$ are uniformly bounded, cumulated depreciation factors $Y_{\xi,\mu}$ are uniformly bounded by above in $D(\xi)$ and $(w^h_\mu)_{\mu\in D(\xi)}$ is uniformly bounded away from zero in $D(\xi)$.*

Proof. Under the conditions of item (i), asset j has a Γ-bubble at a node $\eta\ge\xi$ only if $\lim_{T\to+\infty}\sum_{\mu\in D_T(\eta)}\frac{\gamma_\mu}{\gamma_\eta}\,q_{\mu,j}>0$. This is incompatible with the absence of commodity bubbles in $D(\xi)$. In fact, using the non-arbitrage condition (8), Assumption C together with the particular collateral structure of the mortgage imply that $q_{\mu,j}\le p_\mu C_{\mu,j}\le\sum_{l\in L}p_{\mu,l}Y_\mu(\cdot,l)a_l(\mu^-,\xi)C_{\xi,j,l}$.

Assume that the hypotheses of item (ii) holds. It follows from Theorem 2 that commodities are free of bubbles in $D(\xi)$. Since $(w^h_\mu)_{\mu\geq\xi}$ is uniformly bounded away from zero, it follows from item (iii) in Proposition 1 that, for any $\eta \in D(\xi)$, $\sum_{\eta\in D} \frac{\gamma_\mu}{\gamma^h_\eta}\|p_\mu\|_\Sigma < +\infty$. Thus, independently of τ, assets are free of Γ^h-bubbles due that collateral requirements are uniformly bounded and condition (8) holds. □

It follows from item (i) above that, a bubble in a mortgage loan is always a consequence of a bubble in a commodity that is used as collateral or is part of the real promises. On the other hand, when commodities neither appreciate nor transform into other goods along the event-tree, it follows from item (ii) that, under bounded unitary collateral requirements, well behaved initial endowments assure the absence of price bubbles. In fact, assets will not have a positive price at infinity if the sequence of deflated asset prices is summable, but as this sequence is dominated by the sequence of deflated collateral costs (by non-arbitrage), we just need to have collateral coefficients to be uniformly bounded and deflated commodity prices to be summable (which follows by what is assumed on endowments and depreciation matrices).

In a straightforward extension of our model, we could have allowed for *finite-lived* assets and show that price bubbles would occur if the commodities serving as collateral are priced at infinity. Indeed, the price of a finitely-lived asset will have a bubble if the asset pays in durable goods whose prices have bubbles or if the asset defaults and the surrendered physical collateral is subject to price bubbles.

Appendix A

Following the notation of Section 4, given $(p,q)\in\mathbb{P}$, let $\partial v^h_\xi(z_\xi)$ be the super-differential set of the function v^h_ξ at the point z_ξ. Note that, a vector $(\mathcal{L}'_{\xi,1},\mathcal{L}'_{\xi,2})\in\partial\mathcal{L}^h_\xi(z_\xi,z_{\xi^-},\gamma;p,q)$ if and only there exists $v'_\xi\in\partial v^h_\xi(z_\xi)$ such that both $\mathcal{L}'_{\xi,1}=v'_\xi-\gamma\,\nabla_1\, g^h_\xi(p,q)$ and $\mathcal{L}'_{\xi,2}=-\gamma\,\nabla_2\, g^h_\xi(p,q)$, where $\nabla_1 g^h_\xi(p,q)=(p_\xi,\,q_\xi,\,p_\xi C_{\xi,j}-q_\xi\,)$ and $\nabla_2 g^h_\xi(p,q))=-\left(p_\xi Y_\xi,\,D_\xi(p,q),\,(p_\xi Y_\xi C_{\xi^-,j}-D_{\xi,j}(p,q))_{j\in J}\right).$ Therefore, for any $(\mathcal{L}'_{\xi,1},\mathcal{L}'_{\xi,2})\in\partial\mathcal{L}^h_\xi(z_\xi,z_{\xi^-},\gamma;p,q)$, we have $\mathcal{L}'_{\xi,2}\geq 0$.

Proof of Proposition 1. **(i)** For any $T\in\mathbb{N}$, consider the truncated optimization problem,

$$P^{h,T}_{(p,q)}\qquad \begin{array}{ll} \max & \displaystyle\sum_{\xi\in D^T} v^h_\xi(z_\xi) \\ \text{s.t.} & \left\{\begin{array}{lcl} g^h_\xi\left(z_\xi,z_{\xi^-};p,q\right) & \leq & 0,\quad \forall\xi\in D^T, \\ z_\xi=(x_\xi,\theta_\xi,\varphi_\xi) & \geq & 0,\quad \forall\xi\in D^T,\ \ z_{\xi_0^-}=0. \end{array}\right. \end{array}$$

Note that, there exists a solution for $P^{h,T}_{(p,q)}$ if and only if there is a solution for,

$$\tilde{P}^{h,T}_{(p,q)} \qquad \begin{array}{ll} \max & \sum_{\xi\in D^T} v^h_\xi(z_\xi) \\ \text{s.t.} & \left\{ \begin{array}{lcll} g^h_\xi\left(z_\xi, z_{\xi^-}; p, q\right) & \leq & 0, & \forall \xi \in D^T, \\ z_\xi = (x_\xi, \theta_\xi, \varphi_\xi) & \geq & 0, & \forall \xi \in D^T,\ z_{\xi_0^-} = 0, \\ \theta_{\xi,j} & = & 0, & \forall (\xi, j) \in D^{T-1}\times J \text{ such that } q_{\xi,j} = 0. \end{array}\right. \end{array}$$

Indeed, it follows from the existence of an optimal plan for the consumer problem, giving finite utility, that if $q_{\xi,j} = 0$, for some $(\xi, j) \in D \times J$, then $D_{\mu,j}(p, q) = 0$ for each $\mu \in \xi^+$. Thus, long positions on assets with zero prices do not induce any gains. On the other hand, by Assumption B, commodity prices need to be strictly positive, because we have a finite optimum of individual problem. Also, for any pair $(\xi, j) \in D \times J$, $p_\xi C_{\xi,j} - q_{\xi,j} > 0$, because otherwise individuals may increase their utilities by increasing their loans (detailed arguments, for the case of short-lived assets, are in Araujo, Páscoa and Torres-Martínez (2002, Proposition 1)). Thus, the set of admissible strategies in $\tilde{P}^{h,T}_{(p,q)}$ is compact and, therefore, by the continuity of the utility function we conclude that there is a solution for $\tilde{P}^{h,T}_{(p,q)}$.

Therefore, for any $T \in \mathbb{N}$, the problem $P^{h,T}_{(p,q)}$ has a solution, $(z^{h,T}_\xi)_{\xi\in D^T} = (x^{h,T}_\xi, \theta^{h,T}_\xi, \varphi^{h,T}_\xi)_{\xi\in D}$. It is immediate that $\sum_{\xi\in D^T} v^h_\xi(x^{h,T}_\xi + \sum_{j\in J} C_{\xi,j}\varphi^{h,T}_{\xi,j}) \leq U^h((x^h_\xi + \sum_{j\in J} C_{\xi,j}\varphi^h_{\xi,j})_{\xi\in D})$. Thus, there are non-negative multipliers $(\gamma^{h,T}_\xi)_{\xi\in D^T}$ such that, for each nonnegative plan $(z_\xi)_{\xi\in D^T} \in \mathbb{Z}^{D^T}$, the following saddle point property is satisfied (see Rockafellar (1997), Theorem 28.3),

$$\text{(A.-1)} \qquad \sum_{\xi\in D^T} \mathcal{L}^h_\xi(z_\xi, z_{\xi^-}, \gamma^{h,T}_\xi; p, q) \leq U^h((x^h_\xi + \sum_{j\in J} C_{\xi,j}\varphi^h_{\xi,j})_{\xi\in D}),$$

with $\gamma^{h,T}_\xi g^h_\xi(z^{h,T}_\xi, z^{h,T}_{\xi^-}; p, q) = 0$.

Claim A1. For each $\xi \in D$, the sequence $(\gamma^{h,T}_\xi)_{T\geq t(\xi)}$ is bounded.

Proof. Given $\tilde{D} \subset D$, consider the function $\chi_{\tilde{D}} : D \to \{0, 1\}$ defined by $\chi_{\tilde{D}}(\xi) = 1$ if and only if $\xi \in \tilde{D}$. Given $t \leq T$ and evaluating inequality (A.-1) in $z = (z_\mu)_{\mu\in D^T}$, where $z_\mu = (W^h_\mu, 0, 0)\chi_{D^{t-1}}(\mu)$, we obtain $\sum_{\mu\in D_t} \gamma^{h,T}_\mu p_\mu W^h_\mu \leq U^h(\widehat{x}^h)$. Also, Assumptions A and B imply that, for any $\mu \in D$, both $\min_{l\in L} W^h_{\mu,l}$ and $||p_\mu||_\Sigma$ are strictly positive. Thus, the result follows. ☒

Claim A2. For each $0 < t \leq T$,

$$\text{(A.0)} \qquad 0 \leq -\sum_{\xi\in D_t} \gamma^{h,T}_\xi \nabla_2 g^h_\xi(p, q)\, z^h_{\xi^-} \leq \sum_{\xi\in D\setminus D^{t-1}} v^h_\xi(z^h_\xi),$$

Proof. Given $t \leq T$, if we evaluate (A.-1) in $z = (z_\xi)_{\xi \in D^T}$, with $z_\xi = z_\xi^h \chi_{D^{t-1}}(\xi)$, by budget feasibility of allocation $(z_\xi^h)_{\xi \in D}$, we have

$$-\sum_{\xi \in D_t} \gamma_\xi^{h,T} \nabla_2 g_\xi^h(p,q) \cdot z_{\xi^-}^h + \sum_{\xi \in D^T \setminus D^{t-1}} \gamma_\xi^{h,T} p_\xi w_\xi^h \leq \sum_{\xi \in D \setminus D^{t-1}} v_\xi^h(z_\xi^h)$$

which implies,

$$-\sum_{\xi \in D_t} \gamma_\xi^{h,T} \nabla_2 g_\xi^h(p,q) \cdot z_{\xi^-}^h \leq \sum_{\xi \in D \setminus D^{t-1}} v_\xi^h(z_\xi^h).$$

This concludes the proof, as the left hand side term in the inequality above is non-negative. ⊠

Claim A3. For each $\xi \in D^T \setminus D_T$ and for any plan $y \geq 0$, we have

$$v_\xi^h(y) - v_\xi^h(z_\xi^h) \leq \left(\gamma_\xi^{h,T} \nabla_1 g_\xi^h(p,q) + \sum_{\mu \in \xi^+} \gamma_\mu^{h,T} \nabla_2 g_\mu^h(p,q) \right) (y - z_\xi^h) + \sum_{\eta \in D \setminus D^T} v_\eta^h(z_\eta^h). \tag{A.1}$$

Proof. It follows from (A.-1) that, given $\xi \in D^T \setminus D_T$, for each $y \geq 0$, we can choose a plan $z = (z_\mu)_{\mu \in D^T}$ with $z_\mu = z_\mu^h(1 - \chi_{\{\xi\}}(\mu)) + y\,\chi_{\{\xi\}}(\mu)$, in order to guarantee that,

$$v_\xi^h(y) - \gamma_\xi^{h,T} g_\xi^h(y, z_{\xi^-}^h; p,q) - \sum_{\mu \in \xi^+} \gamma_\mu^{h,T} g_\mu^h(z_\mu^h, y; p,q) \leq v_\xi^h(z_\xi^h) + \sum_{\eta \in D \setminus D^T} v_\eta^h(z_\eta^h). \tag{A.2}$$

Now, as the function $g_\xi^h(\cdot; p,q)$ is affine and the plan $(z_\xi^h)_{\xi \in D} \in B^h(p,q)$, we have that,

$$g_\xi^h(y, z_{\xi^-}^h; p,q) \;=\; \nabla_1 g_\xi^h(p,q) y - p_\xi w_\xi^h + \nabla_2 g_\xi^h(p,q) z_{\xi^-}^h \leq \nabla_1 g_\xi^h(p,q) y - \nabla_1 g_\xi^h(p,q) z_\xi^h,$$

and, for each node $\mu \in \xi^+$,

$$g_\mu^h(z_\mu^h, y; p,q) \;=\; \nabla_1 g_\mu^h(p,q) z_\mu^h - p_\mu w_\mu^h + \nabla_2 g_\mu^h(p,q) y \leq -\nabla_2 g_\mu^h(p,q) z_\xi^h + \nabla_2 g_\mu^h(p,q) y.$$

Substituting the right hand side of inequalities above in equation (A.2) we conclude the proof. ⊠

As the event-tree is countable, Tychonoff Theorem and Claim A1 assure the existence of a common subsequence $(T_k)_{k \in \mathbb{N}} \subset \mathbb{N}$ and non-negative multipliers $(\gamma_\xi^h)_{\xi \in D}$ such that, for each $\xi \in D$, $\gamma_\xi^{h,T_k} \to_{k \to +\infty} \gamma_\xi^h$, and

$$\gamma_\xi^h g_\xi^h(p, q, z_\xi^h, z_{\xi^-}^h) \;=\; 0\,; \tag{A.3}$$

$$\lim_{t \to +\infty} \sum_{\xi \in D_t} \gamma_\xi^h \nabla_2 g_\xi^h(p,q) z_{\xi^-}^h \;=\; 0\,, \tag{A.4}$$

where (A.3) follows from the strictly monotonicity of u_ξ^h, and equation (A.4) is a consequence of Claim A2 (taking first, the limit as T goes to infinity in (A.0) and, afterwards, the limit in t). Moreover, taking the limit as T goes to infinity in (A.1) we obtain that,

$$v_\xi^h(y) - v_\xi^h(z_\xi^h) \leq \left(\gamma_\xi^h \nabla_1 g_\xi^h(p,q) + \sum_{\mu \in \xi^+} \gamma_\mu^h \nabla_2 g_\mu^h(p,q) \right) (y - z_\xi^h), \qquad \forall y \geq 0. \tag{A.5}$$

Therefore, $\gamma_\xi^h \nabla_1 g_\xi^h(p,q) + \sum_{\mu\in\xi^+} \gamma_\mu^h \nabla_2 g_\mu^h(p,q) \in \partial^+ v_\xi^h(z_\xi^h)$, where

$$\partial^+ v_\xi^h(z) := \{v'_\xi \in \mathbb{Z} : \; v_\xi^h(y) - v_\xi^h(z) \leq v'_\xi \cdot (y-z), \quad \forall y \geq 0\}. \tag{A.6}$$

That is, $\partial^+ v_\xi^h(\cdot)$ is the super-differential of the function $v_\xi^h(\cdot) + \delta(\cdot, \mathbb{R}_+^L)$, where $\delta(z,\mathbb{R}_+^L) = 0$, when $z \geq 0$ and $\delta(z,\mathbb{R}_+^L) = -\infty$, in other case. Notice that, for each $z \geq 0$, $\kappa \in \partial\delta(z) \Leftrightarrow 0 \leq k(y-z)$, $\forall y \geq 0$. Thus, by Theorem 23.8 in Rockafellar (1997), for all $z \geq 0$, if $v'_\xi \in \partial^+ v_\xi^h(z)$ then there exists $\tilde{v}'_\xi \in \partial v_\xi^h(z)$ such that both $v'_\xi \geq \tilde{v}'_\xi$ and $(v'_\xi - \tilde{v}'_\xi)\cdot z = 0$. Thus, it follows from (A.5) that there exists, for each $\xi \in D$, a super-gradient $\tilde{v}'_\xi \in \partial v_\xi^h(z_\xi^h)$ such that,

$$\begin{aligned} \gamma_\xi^h \nabla_1 g_\xi^h(p,q) + \sum_{\mu\in\xi^+} \gamma_\mu^h \nabla_2 g_\mu^h(p,q) &\geq \tilde{v}'_\xi, \\ \left(\gamma_\xi^h \nabla_1 g_\xi^h(p,q) + \sum_{\mu\in\xi^+} \gamma_\mu^h \nabla_2 g_\mu^h(p,q)\right) z_\xi^h &= \tilde{v}'_\xi z_\xi^h. \end{aligned}$$

By definition, $\left(\tilde{v}'_\xi - \gamma_\xi^h \nabla_1 g_\xi^h(p,q),\, -\gamma_\xi^h \nabla_2 g_\xi^h(p,q)\right) \in \partial\mathcal{L}_\xi^h(z_\xi^h, z_{\xi^-}^h, \gamma_\xi^h; p, q)$. Therefore, there exists, for each $\xi \in D$, a vector $(\mathcal{L}'_{\xi,1},\, \mathcal{L}'_{\xi,2}) \in \mathcal{L}_\xi^h(z_\xi^h, z_{\xi^-}^h, \gamma_\xi^h; p, q)$ satisfying Euler conditions. Furthermore, the transversality condition is a direct consequence of equation (A.4) jointly with Euler conditions. Indeed,

$$\sum_{\xi\in D_{t-1}} \mathcal{L}'_{\xi,1} z_\xi^h = -\sum_{\xi\in D_t} \mathcal{L}'_{\xi,2} z_{\xi^-}^h = \sum_{\xi\in D_t} \gamma_\xi^h \nabla_2 g_\xi^h(p,q) z_{\xi^-}^h \to_{t\to+\infty} 0.$$

On the other hand, it follows from Euler equations, using the sign property of the Lagrangian, that $\tilde{v}'_\xi - \gamma_\xi^h \nabla_1 g_\xi^h(p,q) \leq 0$. As utility functions u_ξ^h are strictly increasing in the first argument, we know that $\tilde{v}'_\xi$ has a strictly positive first coordinate. Thus, we have that $\gamma_\xi^h\, p_{\xi,1} > 0$, which implies that the multipliers γ_ξ^h are strictly positive, for each $\xi \in D$.

Therefore, there is a plan of Kuhn-Tucker multipliers associated with $(z_\xi^h)_{\xi\in D}$.

(ii) It follows from (EE) that, for each $T \geq 0$,

$$\sum_{\xi\in D^T} \mathcal{L}_\xi^h(z_\xi, z_{\xi^-}, \gamma_\xi^h;\, p, q) - \sum_{\xi\in D^T} \mathcal{L}_\xi^h(z_\xi^h, z_{\xi^-}^h, \gamma_\xi^h;\, p, q) \leq \sum_{\xi\in D_T} \mathcal{L}'_{\xi,1}(z_\xi - z_\xi^h). \tag{A.7}$$

Since, at any node $\xi \in D$ we have that $\gamma_\xi^h g_\xi(z_\xi^h, z_{\xi^-}^h; p, q) = 0$, each $(z_\xi)_{\xi\in D} \in B^h(p,q)$ must satisfy

$$\sum_{\xi\in D^T} u_\xi^h(\widehat{x}_\xi) - \sum_{\xi\in D^T} u_\xi^h(\widehat{x}_\xi^h) \leq \sum_{\xi\in D_T} \mathcal{L}'_{\xi,1}(z_\xi - z_\xi^h).$$

Using the condition (TC) we have that $U^h(\widehat{x}) - U^h(\widehat{x}^h) \leq \limsup_{T\to+\infty} \sum_{\xi\in D_T} \mathcal{L}'_{\xi,1}\, z_\xi$.

Also, Euler conditions imply that $\sum_{\xi\in D_T} \mathcal{L}'_{\xi,1} z_\xi \leq -\sum_{\mu\in D_{T+1}} \mathcal{L}'_{\mu,2}\, z_{\mu^-} \leq 0$, where the last inequality follows from the *sign property* $\mathcal{L}'_{\mu,2} \geq 0$, satisfied at each node of the event-tree. Thus, $U^h(\widehat{x}) \leq U^h(\widehat{x}^h)$, which guarantees that the allocation $(z_\xi^h)_{\xi\in D}$ solves $P_{(p,q)}^h$. Moreover, when

$x^h_\xi + \sum_{j\in J} C_{\xi,j}\varphi^h_{\xi,j} \leq \mathbb{W}_\xi$, for each $\xi \in D$, Assumption B assures that the optimum value is finite.

(iii) As we pointed out in inequality (A.7), the existence of Kuhn-Tucker multipliers $(\gamma^h_\xi)_{\xi\in D}$ implies that, for any $T > 0$, $\sum_{\xi\in D^T} \mathcal{L}^h_\xi(0,0,\gamma^h_\xi;p,q) - \sum_{\xi\in D^T} \mathcal{L}^h_\xi(z^h_\xi, z^h_{\xi^-}, \gamma^h_\xi; p,q) \leq -\sum_{\xi\in D_T} \mathcal{L}'_{\xi,1}\, z^h_\xi$, and, therefore, $\sum_{\xi\in D^T} \gamma^h_\xi p_\xi w^h_\xi \leq U^h(\widehat{x}^h) - \sum_{\xi\in D_T} \mathcal{L}'_{\xi,1}\, z^h_\xi$. Using the transversality condition (TC), we conclude that $\sum_{\xi\in D} \gamma^h_\xi p_\xi w^h_\xi < +\infty$. □

Proof of Claims after Proposition 1. Budget feasibility and Assumption B implies that

$$-\lim_{T\to+\infty} \sum_{\mu\in D_T} \gamma^h_\mu \nabla_2 g^h_\mu(p,q)\, z^h_{\mu^-} = \lim_{T\to+\infty} \sum_{\mu\in D_T} \gamma^h_\mu \nabla_1 g^h_\mu(p,q)\, z^h_\mu - \lim_{T\to+\infty} \sum_{\mu\in D_T} \gamma^h_\mu p_\mu w^h_\mu.$$

Therefore, as deflated endowments are summable, using Euler conditions we assure that our transversality condition is equivalent to $\lim_{T\to+\infty} \sum_{\mu\in D_T} \gamma^h_\mu \nabla_1 g^h_\mu(p,q)\, z^h_\mu = 0$. □

Proof of Corollary 1. Since under prices $(p,q) \in \mathbb{P}$ agent h's problem has a finite optimum, denote by $z^h := (z^h_\xi)_{\xi\in D}$ the optimal plan of agent h at prices (p,q). It follows from Proposition 1-(i) that there is a plan of Kuhn-Tucker multipliers associated with z^h.

Thus, there are $(\mathcal{L}'_{\xi,1}, \mathcal{L}'_{\xi,2})_{\xi\in D} \in \prod_{\xi\in D} \partial\mathcal{L}^h_\xi(z^h_\xi, z^h_{\xi^-}, \gamma^h_\xi; p,q)$ such that, for any $\xi \in D$, $\mathcal{L}'_{\xi,1} + \sum_{\mu\in\xi^+} \mathcal{L}'_{\mu,2} \leq 0$. Using the characterization of $(\mathcal{L}'_{\xi,1}, \mathcal{L}'_{\xi,2})_{\xi\in D}$ at the beginning of this Appendix and the fact that $v'_\xi \in \partial v^h_\xi(z^h_\xi)$ if and only if there is $\alpha_\xi \in \partial u^h_\xi(\widehat{x}^h_\xi)$ such that $v'_\xi = (\alpha_\xi,\, 0,\, (\alpha_\xi C_{\xi,j})_{j\in J})$, we obtain inequalities (5)-(7), as the super gradients of u^h_ξ are vectors with strictly positive entries.

On the other hand, fix $(\xi, j) \in D\times J$. Using the notation introduced after Definition 3, inequalities (5)-(7) imply that, $\eta_\varphi(\Gamma,\xi,j) = \sum_{l\in L} \eta_x(\Gamma,\xi,l) C_{\xi,j,l} - \eta_\theta(\Gamma,\xi,j)$. Therefore, if for each $l \in L$ for which $C_{\xi,j,l} \neq 0$ inequality (5) holds as equality, then $\eta_\theta(\Gamma,\xi,j) = \eta_\varphi(\Gamma,\xi,j) = 0$, which implies that inequalities (6) and (7) holds as equalities. □

Appendix B. Proof of Theorem 1.

An equilibrium for the infinite horizon economy will be found as a limit of equilibria of truncated economies, when the time horizon goes to infinity.

Equilibria in truncated economies. Define, for each $T \in \mathbb{N}$, a truncated economy, $\mathcal{E}^T$, in which agents are restricted to consume and trade assets in the event-tree D^T. Thus, given prices (p,q) in $\mathbb{P}^T := \{(p,q) = (p_\xi, q_\xi)_{\xi\in D^T} \in (\mathbb{R}^L_+ \times \mathbb{R}^J_+)^{D^T} : ||p_\xi||_\Sigma + ||q_\xi||_\Sigma = 1, \quad \forall \xi \in D^T\}$, each agent $h \in H$ has the objective to choose, at each $\xi \in D^T$, a vector $z^{h,T}_\xi = (x^{h,T}_\xi, \theta^{h,T}_\xi, \varphi^{h,T}_\xi) \in \mathbb{Z}$ in order to solve the (truncated) individual problem $P^{h,T}_{(p,q)}$ defined at the beginning of proof of Proposition 1. Now, let $B^{h,T}(p,q)$ be the truncated budget set of agent h in $\mathcal{E}^T$. That is, the set of plans $(z_\xi)_{\xi\in D^T}$ that satisfy the restrictions of problem $P^{h,T}_{(p,q)}$.

An *equilibrium* for $\mathcal{E}^T$ is given by prices $(p^T,q^T)\in\mathbb{P}^T$ jointly with individual plans $z^{h,T}_\xi = (x^{h,T}_\xi,\theta^{h,T}_\xi,\varphi^{h,T}_\xi)_{\xi\in D^T}$ such that: (1) $z^{h,T}$ is an optimal solution for $P^{h,T}_{(p^T,q^T)}$; (2) physical and financial market clear node by node in D^T, in the sense of Definition 1.

Note that, *market feasible allocations*, that is, the non-negative allocations $(x^h_\xi,\theta^h_\xi,\varphi^h_\xi)_{(h,\xi)\in H\times D^T}$ that satisfy market clearing conditions B and C of Definition 1, are bounded in D^T.[7] Therefore, departing from $\mathcal{E}^T$ we can define a *compact economy* $\mathcal{E}^T(K^T)$ by restricting the space of plans of each $h\in H$ to the convex and compact set $K^T := \{z=(x,\theta,\varphi)\in\mathbb{R}^{L\times D^T}_+\times\mathbb{R}^{J\times D^T}_+\times\mathbb{R}^{J\times D^T}_+ : ||z||_\Sigma \le 2\Upsilon^T\}$, which has in its interior the vector Υ^T that is defined as an upper bound for the feasible allocations in D^T.

An equilibrium for $\mathcal{E}(K^T)$ is given by prices $(p^T,q^T)\in\mathbb{P}^T$ and allocations $(z^{h,T}_\xi)_{\xi\in D^T} = (x^{h,T}_\xi,\theta^{h,T}_\xi,\varphi^{h,T}_\xi)_{\xi\in D^T}$, compatible with conditions B and C of Definition 1, such that, for each agent h, the plan $(z^{h,T}_\xi)_{\xi\in D^T}$ solves,

$$(P^{h,T}_{(p^T,q^T)}(K^T)) \qquad \begin{array}{ll} \max & \sum_{\xi\in D^T} v^h_\xi(z_\xi) \\ \text{s.t.} & (z_\xi)_{\xi\in D^T}\in B^{h,T}(p^T,q^T)\cap K^T. \end{array}$$

If we assure the existence of equilibrium for $\mathcal{E}^T(K^T)$, the economy $\mathcal{E}^T$ has also an equilibrium, given that optimal allocation of $\mathcal{E}^T(K^T)$ will be, by construction, interior points of set K^T, budget sets are convex and utility functions are concave under Assumption B.

Generalized Games. To prove the existence of equilibrium in $\mathcal{E}^T(K^T)$ we introduce a game $\mathcal{G}^T$, where each $h\in H$ takes prices $(p,q)\in\mathbb{P}^T$ as given and solves the compact truncated problem above. Moreover, associated to each node in D^T there is an auctioneer who, given plans $(z^h_\xi)_{(h,\xi)\in H\times D^T}\in\prod_{h\in H} K^T$ has the objective to find prices $(p_\xi,q_\xi)\in\Delta^{L+J-1}_+$ in order to maximize the function,

$$\text{(B.1)}\quad p_\xi\sum_{h\in H}\left(x^h_\xi+\sum_{j\in J}C_{\xi,j}\varphi^h_{\xi,j}-w^h_\xi-Y_\xi x^h_{\xi^-}-Y_\xi\sum_{j\in J}C_{\xi^-,j}\varphi^h_{\xi^-,j}\right)+\sum_{j\in J}q_{\xi,j}\sum_{h\in H}\left(\theta^h_{\xi,j}-\varphi^h_{\xi,j}\right)$$

where, $z^h_\xi = (x^h_\xi,\theta^h_\xi,\varphi^h_\xi)$ and, for convenience of notations, for each $(h,j)\in H\times J$ we put $(x^h_{\xi_0^-},\theta^h_{\xi_0^-,j},\varphi^h_{\xi_0^-,j}) = (0,0,0)$ and $(\widehat{C}_{\xi_0,j},Y_{\xi_0}) = (0,0)$, for all $j\in J$

[7]Indeed, autonomous consumption allocations, $(x^h_\xi)_{(h,\xi)\in H\times D^T}$ are bounded by above, node by node, by the aggregated physical endowments. The short-sales $(\varphi^h_{\xi,j})_{(h,\xi)\in H\times D^T}$ are bounded, at each $\xi\in D^T$, by $\sum_{l\in L}\mathbb{W}_{\xi,l}$ divided by the positive number $\|C_{\xi,j}\|_\Sigma$. Thus, long positions $(\theta^h_{\xi,j})_{(h,\xi)\in H\times D^T}$ are also bounded, because are less than or equal to the aggregated short sales.

A vector $\Big[(p^T, q^T); (z_\xi^{h,T})_{h\in H}\Big]_{\xi\in D^T}$ that solves simultaneously the problems above is called a *(Cournot-Nash) equilibrium* of $\mathcal{G}^T$.

LEMMA B1. For each $T \in \mathbb{N}$ there is an equilibrium for $\mathcal{G}^T$.

PROOF. The objective function of each participant in the game is continuous and quasi-concave in the own strategy. For auctioneers, the correspondences of admissible strategies are continuous, with non-empty, convex and compact values. Also, the budget restriction correspondence of each agent, $(p, q) \twoheadrightarrow B^{h,T}(p, q) \cap K^T$, has non-empty, convex and compact values. Therefore, in order to find an equilibrium of the generalized game (as a fixed point of the set function given by the product of optimal strategies correspondences), it is sufficient to prove that budget set correspondences are continuous.

The upper hemi-continuity follows from compact values and closed graph properties, that are a direct consequence of continuity of functions g_ξ^h. Thus, the main difficulty resides in showing the lower hemi-continuity property. Now, as for each price $(p, q) \in \mathbb{P}^T$ the set $B^{h,T}(p, q) \cap K^T$ is convex and compact, it is sufficient to assure that the (relative) interior correspondence $(p, q) \twoheadrightarrow$ $int(B^{h,T}(p, q)) \cap K^T$ has non-empty values. But this last property follows from Assumption A. In fact, cumulated endowments are such that $W_\xi^h \gg 0$, for each $h \in H$, and, therefore, given any plan of prices $(p, q) \in \mathbb{P}^T$, the plan $(\tilde{x}_\xi^h; \tilde{\theta}_\xi^h; \tilde{\varphi}_\xi^h)_{\xi\in D^T} := \left(\frac{W_\xi^h}{2^{t(\xi)+1}} - \sum_{j\in J} C_{\xi,j}\epsilon_\xi^h;\ 0;\ \epsilon_\xi(1, 1, \ldots, 1)\right)_{\xi\in D^T}$, where for each $\xi \in D^T$, $\epsilon_\xi^h = \min_{(l,\mu)\in L\times\xi^+} \left\{\frac{W_{\xi,l}^h}{2^{t(\xi)+2}\left(1+\sum_{j\in J}(C_{\xi,j})_l\right)};\ \frac{W_{\mu,l}^h}{2^{t(\xi)+2}\left(1+\sum_{j\in J} Y_\mu(l,\cdot)C_{\xi,j}\right)}\right\}$, is budget feasible and belongs to the relative interior of the set $B^{h,T}(p, q) \cap K$. □

LEMMA C2. For each $T \in \mathbb{N}$ there is an equilibrium for $\mathcal{E}^T(K^T)$.

PROOF. We know that there exists an equilibrium for $\mathcal{G}^T$, namely $\Big[(p^T, q^T); (z_\xi^{h,T})_{h\in H}\Big]_{\xi\in D^T}$. By definition, each agent $h \in H$ solves problem $P^{h,T}(K^T)$ by choosing the plan $(z_\xi^{h,T})_{\xi\in D^T}$. Thus, it is sufficient to verify, for each node $\xi \in D^T$, the validity of conditions B and C of Definition 1.

As budget feasibility implies that $\sum_{h\in H} g_\xi^h(z_\xi^{h,T}, z_{\xi^-}^{h,T}, p^T, q^T) \le 0$, the optimal value of auctioneers objective functions is less than or equal to zero. This implies that conditions B and C of Definition 1 are satisfied as inequalities. That is, there does not exist excess demand in physical and financial markets.

Thus, as the individual demands for commodities or assets are bounded by the aggregate supply of resources, the optimal bundles that were chosen by the agents are interior points of K^T. Therefore, monotonicity of utility function implies that, for each $\xi \in D^T$, $\sum_{h\in H} g_\xi^h(z_\xi^{h,T}, z_{\xi^-}^{h,T}, p^T, q^T) = 0$. In other words, Walras' law holds.

The existence of an optimal solution for $P^{h,T}(K^T)$ in the interior of the set K^T implies that $p_\xi^T \gg 0$ and, therefore, condition B of Definition 1 holds, as a direct consequence of Walras' law, strictly positive commodity prices and the absence of excess demand in physical markets. By analogous arguments, condition C of Definition 1 holds, at a node $\xi \in D^T$, for those assets $j \in J$ which have a strictly positive price $q_{\xi,j}^T > 0$.

Given $\xi \in D^T$, denote by $\tilde{J}_\xi \subset J$ the set of assets with zero price at ξ and let $\Delta(\theta_\xi^T, \theta_{\xi^-}^T)_{\xi,j}$ be the excess demand of asset j at node ξ, associated with long positions $(\theta_\xi^T, \theta_{\xi^-}^T) = (\theta_\xi^{h,T}, \theta_{\xi^-}^{h,T})_{h\in H}$ (it follows from previous arguments that $\Delta(\theta_\xi^T, \theta_{\xi^-}^T)_{\xi,j} \leq 0$). If $j \in \tilde{J}_\xi$, then optimality of agents' allocations assures that the asset does not deliver any payment at the successor nodes $\mu \in \xi^+$ (if this nodes are in D^T). Therefore, if we change the portfolio allocation $(\theta_\xi^{h,T})_{h\in H}$ to $\tilde{\theta}_\xi^{h,T} = \theta_\xi^{h,T} - \frac{1}{\#H}\Delta(\theta_\xi^T, \theta_{\xi^-}^T)_{\xi,j}$, we assure that, at node ξ, and for asset j, condition D holds. Moreover, the new allocation is budget feasible, optimal, and we do not lose the market clearing condition in physical markets at node $\mu \in \xi^+$.

However, the total supply of asset j at nodes $\mu \in \xi^+$ can change. Therefore, in order to apply the trick above, node by node, asset by asset, to obtain an optimal allocation that satisfies Condition D for each asset, it is sufficient to prove that, after changing portfolios at a node ξ, the new excess demand, at nodes $\mu \in \xi^+$, $\Delta(\theta_\mu^T, \tilde{\theta}_\xi^T)_{\mu,j}$ is still less than or equal to zero and that $\Delta(\theta_\mu^T, \tilde{\theta}_\xi^T)_{\mu,j}$ can be negative only for assets in $\tilde{J}_\mu$.

Fix $j \in \tilde{J}_\xi$. It follows by the definition of $\tilde{\theta}_\xi^{h,T}$ that, at any $\mu \in \xi^+$, $\Delta(\theta_\mu^T, \tilde{\theta}_\xi^T)_{\mu,j} \leq \Delta(\theta_\mu^T, \theta_\xi^T)_{\mu,j}$. Now, as at each $\mu \in \xi^+$, $D_{\mu,j}(p^T, q^T) = 0$ then asset j defaults at nodes $\mu \in \xi^+$. Therefore, $(\lambda_{\mu,j}^T)_{\mu\in\xi^+} = 0$ and $(\Delta(\theta_\mu^T, \tilde{\theta}_\xi^T)_{\mu,j})_{\mu\in\xi^+} = (\Delta(\theta_\mu^T, \theta_\xi^T,)_{\mu,j})_{\mu\in\xi^+}$, which concludes the proof. □

In the previous lemma we found an equilibrium for $\mathcal{E}^T(K^T)$. It is not difficult to verify that this equilibrium constitutes also an equilibrium for $\mathcal{E}^T$.

Asymptotic equilibria. For each $T \in \mathbb{N}$, fix an equilibrium $\left[(p^T, q^T); (z_\xi^{h,T})_{h\in H}\right]_{\xi\in D^T}$ of $\mathcal{E}^T$. We know that there exist non-negative multipliers $(\gamma_\xi^{h,T})_{\xi\in D^T}$ such that, $\gamma_\xi^{h,T} g_\xi^h(z_\xi^{h,T}, z_{\xi^-}^{h,T}; p, q) = 0$, and the following saddle point property is satisfied, for each nonnegative plan $(z_\xi)_{\xi\in D^T}$ (see Rockafellar (1997), Section 28, Theorem 28.3),

$$\sum_{\xi\in D^T} \mathcal{L}_\xi^h(z_\xi, z_{\xi^-}, \gamma_\xi^{h,T}; p^T, q^T) \leq \sum_{\xi\in D^T} v_\xi^h(z_\xi^{h,T}). \tag{B.2}$$

As $v^h_\xi(z^{h,T}_\xi) \leq u^h_\xi(\mathbb{W}_\xi)$, analogously to Claim A1 in Appendix A, for each $\xi \in D$ and for all $T \geq t(\xi)$,

$$0 \leq \gamma^{h,T}_\xi < \frac{U^h(\mathbb{W})}{\underline{W}^h_\xi \, \|p^T_\xi\|_\Sigma}. \tag{B.3}$$

where $\underline{W}^h_\xi = \min_{l\in L} W^h_{\xi,l} > 0$.

Lemma C3. For each $\xi \in D$, there is a strictly positive lower bound for $(\|p^T_\xi\|_\Sigma)_{T>t(\xi)}$.

Proof. Given $\xi \in D$ and $T > t(\xi)$, optimality of $z^{h,T}$ in $P^{h,T}_{(p^T,q^T)}$ implies that $p^T_\xi C_{\xi,j} \geq q^T_{\xi,j}$, for each $j \in J$. Thus, for each $j \in J$, there is $\overline{m}_{\xi,j} > 0$ such that, $q^T_{\xi,j} \leq \overline{m}_{\xi,j}\|p^T_\xi\|_\Sigma$. Adding in j, we obtain that $\|q^T_\xi\|_\Sigma \leq \|p^T_\xi\|_\Sigma \sum_{j\in J} \overline{m}_{\xi,j}$. Finally, as $\|q^T_\xi\|_\Sigma = 1 - \|p^T_\xi\|_\Sigma$, at each node $\xi \in D$, independently of T, $\|p^T_\xi\|_\Sigma \geq \frac{1}{1+\sum_{j\in J}\overline{m}_{\xi,j}} > 0$. □

Therefore, the sequence $\left[(p^T_\xi, q^T_\xi); (z^{h,T}_\xi, \gamma^{h,T}_\xi)_{h\in H}\right]_{T>t(\xi)}$ is bounded. Applying Tychonoff Theorem we find, as in the proof of Proposition 1, a subsequence $(T_k)_{k\in\mathbb{N}} \subset \mathbb{N}$ such that, for each $\xi \in D$, $\left[(p^{T_k}_\xi, q^{T_k}_\xi); (z^{h,T_k}_\xi, \gamma^{h,T_k}_\xi)_{h\in H}\right]_{T_k>t(\xi)}$ converges, as k goes to infinity, to an allocation $\left[(\overline{p}_\xi, \overline{q}_\xi); (\overline{z}^h_\xi, \overline{\gamma}^h_\xi)_{h\in H}\right]$.

Moreover, the limit allocations $\left[(\overline{z}^h_\xi)_{\xi\in D}\right]_{h\in H}$ are budget feasible at prices $(\overline{p}, \overline{q}) \in \mathbb{P}$, and satisfy market feasibility conditions at each node in the event-tree. Thus, in order to assure that $\left[(\overline{p}_\xi, \overline{q}_\xi); (\overline{z}^h_\xi, \overline{\gamma}^h_\xi)_{h\in H}\right]_{\xi\in D}$ is an equilibrium we just need, by the results of Section 4, to verify that, for each agent $h \in H$, $(\overline{z}^h_\xi, \overline{\gamma}^h_\xi)_{\xi\in D}$ satisfies Euler and transversality conditions.

Lemma C4. For each $t > 0$ we have that,

$$0 \leq -\sum_{\xi\in D_t} \overline{\gamma}^h_\xi \nabla_2 g^h_\xi(\overline{p}, \overline{q}) \cdot \overline{z}^h_{\xi^-} \leq \sum_{\xi\in D\setminus D^{t-1}} v^h_\xi(\overline{z}^h_\xi), \tag{B.4}$$

Moreover, for each $\xi \in D$ and for all plan $y \geq 0$, we have that

$$v^h_\xi(y) - v^h_\xi(\overline{z}^h_\xi) \leq \left(\overline{\gamma}^h_\xi \nabla_1 g^h_\xi(\overline{p}, \overline{q}) + \sum_{\mu\in\xi^+} \overline{\gamma}^h_\mu \nabla_2 g^h_\mu(\overline{p}, \overline{q})\right) \cdot (y - \overline{z}^h_\xi). \tag{B.5}$$

Proof. The proof is analogous to those made in Claims A2 and A3 (Appendix A), changing prices (p, q) by (p^T, q^T), and taking the limit as T goes to infinity. □

Thus, since $\sum_{\xi\in D\setminus D^{t-1}} v^h_\xi(\overline{z}^h_\xi) \leq \sum_{\xi\in D\setminus D^{t-1}} u^h_\xi(\mathbb{W}_\xi)$, we have $\lim_{t\to+\infty} \sum_{\xi\in D_t} \overline{\gamma}^h_\xi \nabla_2 g^h_\xi(\overline{p}, \overline{q})\overline{z}^h_{\xi^-} = 0$. Moreover, $\left(\overline{\gamma}^h_\xi \nabla_1 g^h_\xi(\overline{p}, \overline{q}) + \sum_{\mu\in\xi^+} \overline{\gamma}^h_\mu \nabla_2 g^h_\mu(\overline{p}, \overline{q})\right) \in \partial^+ v^h(\overline{z}^h_\xi)$. By the same arguments made in the proof of Proposition 1-(i) (see Appendix A) we conclude that Euler equations and transversality conditions hold. Therefore, it follows from Proposition 1-(ii) that the allocation $(\overline{z}^h_\xi)_{\xi\in D}$ is optimal for agent $h \in H$, which concludes the proof of the Theorem 1.

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