Core equivalence and welfare properties without divisible goods *

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First version November 2001, this version May 2005

Abstract

We study an economy where all goods entering preferences or production processes are indivisible. Fiat money not entering consumers' preferences is an additional perfectly divisible parameter. We establich a First and Second Welfare Theorem and a core equivalence result for the rationing equilibrium concept introduced in Florig and Rivera (2005a). The rationing equilibrium can be considered as a natural extension of the Walrasian notion when all goods are indivisible at the individual level but perfectly divisible at the level of the entire economy.

As a Walras equilibrium is a special case of a rationing equilibrium, our results also hold for Walras equilibria.

Keywords: indivisible goods, competitive equilibrium, Pareto optimum, core.

JEL Classification: D50, D60

1 Introduction

In general equilibrium theory it is well known that a Walrasian equilibrium may fail to exist in the presence of indivisible goods (Henry (1970)) and even the core may be empty (Shapley and Scarf (1974)).

In order to consider the presence of indivisible goods in the economy, numerous authors as Broome (1972), Mas-Colell (1977), Khan and Yamazaki (1981), Quinzii (1984) - see Bobzin (1998) for a survey - consider economies with indivisible commodities and one perfectly divisible good. All these contributions suppose that the divisible

*This work was supported by FONDECYT - Chile, Project nr. 1000766-2000, ECOS and ICM Sistemas Complejos de Ingeniería.
We thank Yves Balasko and Olivier Gossner for helpful comments.

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commodity satisfies overriding desirability, i.e. it is so desirable by the agents that it can replace the consumption of indivisible goods. Moreover, every agent initially owns an important quantity of this good. In such case, the non-emptiness of the core and existence of a Walras equilibrium is then ensured.

In the model developed in Florig and Rivera (2005a) it is assumed that all the consumption goods are indivisible at individual level but perfectly divisible at the aggregate ecoonmy. The presence of a parameter, called fiat money, that does not participate in the preferences and whose only role is to facilitate the exchange among individuals helps us to demonstrate the existence of a competitive equilibrium called *rationing equilibrium*. Under additional assumptions on the distribution of fiat money it can be proved that the rationing equilibrium is a Walras equilibrium. Moreover, in a parallel paper (Florig and Rivera (2005b)) we prove that the rationing equilibrium converges to a Walrasian one when the level of indivisibility converges to zero. Thus, the rationing equilibrium concept appears as a natural extension of the Walras equilibrium in the framework mentioned.

Here we study welfare properties and core equivalence for rationing equilibria.

In our context preference relations are always locally satiated since all goods are indivisible. Konovalov (2005) shows that the standard core concepts have undesirable properties in economies with satiation. He introduces the rejective core which overcomes such drawbacks.

Using the blocking concept introduce by Konovalov (2005), we show in Proposition 3.1 that a rationing equilibrium cannot be blocked, whereas in Proposition 3.2 is proven that a rejective core allocation can be decentralized as a Walras equilibrium by an appropriate redistribution of fiat money.

With respect to the welfare analysis, we point out that at a rationing equilibrium (and for Walras one in our setting as well) it is possible that some consumers may own commodities which are worthless to them as a consumption good (or they own more than some satiation level). With indivisible goods, the value of these commodities at the equilibrium may be so small that selling them does not enable to buy more of the goods they are interested. Thus, they may waste these commodities, which may however be very useful and expensive for other agents. So the market is not as efficient as in the standard Arrow-Debreu setting (Arrow and Debreu (1954)). However, even though the standard notion of “strong Pareto optimality” fails for our equilibrium notion, this is not the case when instead we consider “weak Pareto” optimality. As e.g. Florig (2002), we use a slightly different notion of weak Pareto optimality than the one usually encountered in the literature. Using the standard weak Pareto optimality would imply that in the presence of a consumer not interested in any good, all feasible allocations are weakly Pareto optimal. We avoid this drawback.

2 Model and preliminaries

We set $L \equiv \{1, \ldots, L\}$, $I \equiv \{1, \ldots, I\}$ and $J \equiv \{1, \ldots, J\}$ to denote the finite set of commodities, the finite sets of types of consumers and producers, respectively. We assume that each type $k \in I, J$ of agents consists of a continuum of identical individuals indexed by a set $T_k \subset I\!R$ of finite Lebesgue measure[1]. We set $\mathcal{I} = \cup_{i\in I} T_i$ and $\mathcal{J} = \cup_{j\in J} T_j$. Of course, $T_k \cap T_{k'} = \emptyset$ if $k \neq k'$. Given $t \in \mathcal{I}$ ($\mathcal{J}$), let

$$i(t) \in I \ (j(t) \in J)$$

be the index such that $t \in T_{i(t)}$ ($t \in T_{j(t)}$).

Each firm of type $j \in J$ is characterized by a finite production set[2] $Y_j \subset I\!R^L$ and the aggregate production set of firms of type $j \in J$ is the convex hull of $\mathcal{L}(T_j)Y_j$, which is denoted by $\mathrm{co}\,[\mathcal{L}(T_j)Y_j]$.

Every consumer of type $i \in I$ is characterized by a finite consumption set $X_i \subset R^L$, an initial endowment $e_i \in I\!R^L$ and a strict preference correspondence $P_i : X_i \to X_i$.

Let $e = \sum_{i\in I} \mathcal{L}(T_i)e_i$ be the aggregate initial endowment of the economy and for $(i,j) \in I \times J$, $\theta_{ij} \geq 0$ is the share of type $i \in I$ consumers in type $j \in J$ firms. For all $j \in J$, assume that $\sum_{i\in I} \mathcal{L}(T_i)\theta_{ij} = 1$.

The initial endowment of fiat money for an individual $t \in \mathcal{I}$ is defined by $m(t)$, where $m : \mathcal{I} \to I\!R_+$ is a Lebesgue-measurable and bounded mapping.

Given all the above, an *economy* $\mathcal{E}$ is a collection

$$\mathcal{E} = \Big((X_i, P_i, e_i, m)_{i\in I}, (Y_j)_{j\in J}, (\theta_{ij})_{(i,j)\in I\times J}\Big),$$

an *allocation* (or consumption plan) is an element of

$$X = \Big\{x \in \mathcal{L}^1(\mathcal{I}, \cup_{i\in I} X_i) \mid x_t \in X_{i(t)} \text{ for a.e. } t \in \mathcal{I}\Big\},$$

a *production plan* is an element of

$$Y = \Big\{y \in \mathcal{L}^1(\mathcal{J}, \cup_{j\in J} Y_j) \mid y_t \in Y_{j(t)} \text{ for a.e. } t \in \mathcal{J}\Big\},$$

and the *feasible consumption-production plans* are elements of

$$A(\mathcal{E}) = \Big\{(x,y) \in X \times Y \mid \int_{\mathcal{I}} x_t = \int_{\mathcal{J}} y_t + e\Big\}.$$

In the rationing equilibrium definition below we will employ *pointed cones* in $I\!R^L$, which is the set of convex cones $\mathcal{C} \subseteq I\!R^L$ such that $K \in \mathcal{C}$ if and only if $-K \cap K = \{0_{I\!R^L}\}$.

Given $p \in I\!R^L_+$, let us define the *supply* and *profit* of a type $j \in J$ firm as

$$S_j(p) = \mathrm{argmax}_{y\in Y_j}\ p\cdot y \qquad\qquad \pi_j(p) = \mathcal{L}(T_j)\sup_{y\in Y_j}\ p\cdot y$$

and given additionally $K \in \mathcal{C}$ we define the rationing supply (in the following simply *supply*) for a firm $t \in \mathcal{J}$ by

$$\sigma_t(p,K) = \{y \in S_{j(t)}(p) \mid (Y_{j(t)} - y) \cap p^{\perp} \subset -K\}.$$

[1]Without loss of generality we may assume that T_k is a compact interval of $I\!R$. In the following, we note by $\mathcal{L}(T_k)$ the Lebesgue measure of set $T_k \subseteq I\!R$. Finally, we denote by $\mathcal{L}^1(A,B)$ the Lebesgue integrable functions from $A \subset I\!R$ to $B \subset I\!R^L$.

[2]That is, the number of admissible production plans for the firm is finite.

For prices $(p,q) \in I\!R^L \times I\!R_+$, we denote the *budget set* of a consumer $t \in \mathcal{I}$ by $B_t(p,q) = \left\{x \in X_{i(t)} \,|\, p \cdot x \leq w_t(p,q)\right\}$ where $w_t(p,q) = p \cdot e_{i(t)} + qm(t) + \sum_{j \in J} \theta_{i(t)j}\pi_j(p)$ is the wealth of individual $t \in \mathcal{I}$. The set of *maximal elements* for the preference relation in the budget set for consumer $t \in \mathcal{I}$ is denoted by $d_t(p,q)$ and given that, we define the weak demand at the respective prices as

$$D_t(p,q) = \limsup_{(p',q') \to (p,q)} d_t(p',q').$$

Previous auxiliary concept is used to define our notion of *demand*, which for a cone $K \in \mathcal{C}$ and prices $(p,q) \in I\!R^L \times I\!R_+$ is defined as

$$\delta_t(p,q,K) = \{x \in D_t(p,q) \,|\, (P_t(x) - x) \cap p^\perp \subset K\}.$$

Remark 2.1 An economic interpretation of weak demand and the demand is given in Florig and Rivera (2005a). There it is proven that if $qm_t > 0$[3] then

$$D_t(p,q) = \{x \in B_t(p,q) \mid p \cdot P_{i(t)}(x) \geq w_t(p,q),\ x \notin \text{co}P_{i(t)}(x)\}.$$

With the previous concepts, we can now define our equilibrium notions.

Definition 2.1 Let $(x,y,p,q) \in A(\mathcal{E}) \times I\!R^L \times I\!R_+$ and $K \in \mathcal{C}$.

We call (x,y,p,q) a *Walras equilibrium* with money of $\mathcal{E}$ if for a.e. $t \in \mathcal{I}$, $x_t \in d_t(p,q)$ and for a.e. $t \in \mathcal{J}$, $y_t \in S_{j(t)}(p)$.

We call (x,y,p,q,K) a *rationing equilibrium* of $\mathcal{E}$ if for a.e. $t \in \mathcal{I}$, $x_t \in \delta_t(p,q,K)$ and for a.e. $t \in \mathcal{J}$, $y_t \in \sigma_t(p,K)$.

Remark 2.2

a.- Note that every Walras equilibrium is a rationing equilibrium. We refer to Florig and Rivera(2005a) for the conditions that ensure existence of these two equilibrium notions in the current framework.

b.- It is well known that a Walras equilibrium may fail to exists when goods are indivisible. Mathematically this comes from the fact that in our framework the correspondence d_i is not necessarily upper semi continuous with respect to (p,q), unlike the regularized notion of it (D_i).

(c) It may be worthwhile to mention that the term $p^\perp$ can be dropped in the definition of demand and supply. The resulting equilibrium notion coincide with the rationing equilibrium concept as given. The term $p^\perp$ makes it however easier to check whether an allocation is a rationing equilibrium or not.

[3] In that case, paper money can be used as an intermediary good. In the contrary case, due to paper money not entering consumers preferences, it could dropped from the economy without further consequences.

3 Core properties

Konovalov (2005) shows that the standard core notions have undesirable properties when preferences are satiated, which is obviously our case due the indivisibility of all goods in our setting. To overcome these undesirable properties, he proposes a new notion of blocking that is used here to study the core properties of the rationing equilibrium. Thus, we establish equivalence between rationing equilibrium allocations (which satisfy $qm(t) > 0$ a.e.) and the rejective core, and then we illustrate our result with examples.

The following definition is an straightforward extension of Konovalov's (2005) rejective core to our setting.

Definition 3.1 The coalition $T \subset \mathcal{I}$ **rejects** $(x, y) \in A(\mathcal{E})$, if there exist a measurable partition U, V of T, and an allocation $(x', y') \in A(\mathcal{E})$ such that the following holds

(i)

$$\int_T x'_t dt = \int_U \left[x_t + \sum_{j \in J} \theta_{i(t)j} \int_{T_j} (y'_\tau - y_\tau) d\tau \right] dt + \int_V \left[e_{i(t)} + \sum_{j \in J} \theta_{i(t)j} \widetilde{y}_j(V) \right] dt,$$

with $\widetilde{y}_j(V) = \int_V y_\tau d\tau$,

(ii) $x'_t \in P_{i(t)}(x_t)$ for a.e. $t \in T$,

(iii) for a.e. $t \in \mathcal{I} \setminus T$,

$$\left[e_{i(t)} + \sum_{j \in J} \theta_{i(t)j} \mathcal{L}(T_j) Y_j \right] \cap P_{i(t)}(x'_t) = \emptyset.$$

The **rejective core** $\mathcal{RC}(\mathcal{E})$ of $\mathcal{E}$ is the set of $(x, y) \in A(\mathcal{E})$ that cannot be rejected by a non-negligible coalition.

Proposition 3.1 Let (x, y, p, q, K) be a rationing equilibrium such that for a.e. $t \in \mathcal{I}$, $qm(t) > 0$, then $(x, y) \in \mathcal{RC}(\mathcal{E})$.

Proof. Let $T \subset \mathcal{I}$ with $\mathcal{L}(T) > 0$, U, V a measurable partition of T and $(x', y') \in A(\mathcal{E})$ such that conditions (i)-(iii) of Definition 3.1 hold.

By condition (ii) and Remark 2.1 we have that for a.e. $t \in T$, $p \cdot x'_t \geq w_t(p, q)$, and then, considering that $qm(t) > 0$ for a.e. $t \in \mathcal{I}$, we conclude that for a.e. $t \in \mathcal{I}$

$$p \cdot \left[e_{i(t)} + \sum_{j \in J} \theta_{i(t)j} \widetilde{y}_j(V) \right] < w_t(p, q) \text{ for a.e } t \in V.$$

On the other hand, by profit maximization we have that for a.e. $t \in U$

$$p \cdot \sum_{j \in J} \theta_{i(t)j} \int_{T_j} (y'_\tau - y_\tau) d\tau \leq 0,$$

and therefore

$$p \cdot \left[x_t + \sum_{j \in J} \theta_{i(t)j} \int_{T_j} (y'_\tau - y_\tau) d\tau \right] \leq w_t(p,q).$$

Hence, if $\mathcal{L}(V) > 0$ we would have

$$p \cdot \int_T x'_t dt \geq \int_T w_t(p,q) dt >$$

$$p \cdot \left[\int_U [x_t + \sum_{j \in J} \theta_{i(t)j} \int_{T_j} (y'_\tau - y_\tau) d\tau] dt + \int_V [e_{i(t)} + \sum_{j \in J} \theta_{i(t)j} \widetilde{y}_j(V)] dt \right]$$

contradicting condition (i). So we have $\mathcal{L}(V) = 0$ and we must have

$$p \cdot \int_U x'_t dt = p \cdot \left[\int_U [x_t + \sum_{j \in J} \theta_{i(t)j} \int_{T_j} (y'_\tau - y_\tau) d\tau \right] dt.$$

By Remark 2.1 for a.e. $t \in U$, $p \cdot [x'_t - x_t] \geq 0$ and since by profit maximization $p \cdot \sum_{j \in J} \theta_{i(t)j} \int_{T_j} (y'_\tau - y_\tau) d\tau \leq 0$, we must have for a.e. $t \in U$, $p \cdot [x'_t - x_t] = 0$ and $p \cdot \sum_{j \in J} \theta_{i(t)j} \int_{T_j} (y'_\tau - y_\tau) d\tau = 0$. Therefore by definition of demand and supply for a.e. $t \in U$, $[x'_t - x_t] \in K$ and $\sum_{j \in J} \theta_{i(t)j} \int_{T_j} (y'_\tau - y_\tau) d\tau \in -K$ and integrating over U, we have $\int_U (x'_t - x_t) dt \in K$ and $\int_U [\sum_{j \in J} \theta_{i(t)j} \int_{T_j} (y'_\tau - y_\tau) d\tau] dt \in -K$. Now, since

$$K \ni \int_U (x'_t - x_t) dt = \int_U \left[\sum_{j \in J} \theta_{i(t)j} \int_{T_j} (y'_\tau - y_\tau) d\tau \right] dt \in -K$$

we have $\int_U (x'_t - x_t) dt \in -K \cap K = 0_{\mathbb{R}^L}$. Since for a.e. $t \in U$, $(x'_t - x_t) \in K$, this implies that for a.e. $t \in U$, $x'_t = x_t$, which is a contradiction with condition (ii). □

With production, the absence of local non-satiation entails the possible existence of rejective core allocations that can not be decentralized. This is due to the fact that a consumer at a satiation point does not care whether a firm he entirely owns chooses a profit maximizing production plan or not. This could be overcome by a refinement of profit maximization as in (Florig 2001). Instead, we show that without a production sector every rejective core allocation can be decentralized.

Proposition 3.2 Suppose $J = \emptyset$ (exchange economy). Then, for every $x \in \mathcal{RC}(\mathcal{E})$ there exists $(p, m') \in \mathbb{R}^L \setminus \{0\} \times \mathcal{L}^1(\mathcal{I}, \mathbb{R}_{++})$ such that $(x, p, q = 1)$ is a Walras equilibrium with money of the economy $\mathcal{E}$ when replacing m by m'.

Proof. Let $x \in \mathcal{RC}(\mathcal{E})$. Since the number of types is finite and the consumption sets are finite, we can define a finite set of consumer types $A \equiv \{1, \ldots, A\}$ satisfying the following

(i) $(T_a)_{a \in A}$ is a finer partition of $\mathcal{I}$ than $(T_i)_{i \in I}$,

(ii) for every $a \in A$, there exists x_a such that for every $t \in T_a$, $x_t = x_a$.

Set

$$H_a = \mathcal{L}(T_{i(a)}) \left[\mathrm{co}P_{i(a)}(x_a) - x_a\right], \ G_a = \mathcal{L}(T_{i(a)}) \left[\mathrm{co}P_{i(a)}(x_a) - e_{i(a)}\right],$$

$$\mathcal{K} = \mathrm{co}\left[\cup_{a\in A}(G_a \cup H_a)\right].$$

As a first step of the demonstration we will prove that $0 \notin \mathcal{K}$. Otherwise there exist $(\lambda_a) \in [0,1]^A$ and $(\mu_a) \in [0,1]^A$ with $\sum_{a\in A}(\lambda_a + \mu_a) = 1$ and $\xi_a \in \mathrm{co}P_{i(a)}(x_a)$ for all $a \in A$, such that

$$\sum_{a\in A} \left[\lambda_a \mathcal{L}(T_a)(\xi_a - x_a) + \mu_a \mathcal{L}(T_a)(\xi_a - e_a)\right] = 0.$$

Thus there exists a measurable partition U, V of T, $\xi \in X$ such that for a.e. $t \in T$ and $\xi_t \in P_{i(t)}(x_t)$ and for all $a \in A$

$$\mathcal{L}(U \cap T_a) = \frac{1}{2}\lambda_a \mathcal{L}(T_a)$$

$$\mathcal{L}(V \cap T_a) = \frac{1}{2}\mu_a \mathcal{L}(T_a).$$

and it is easy to check that

$$\int_T \xi_t = \int_U x_t + \int_V e_t.$$

Now define $\zeta \in X$ by

$$\zeta_t = \begin{cases} \xi_t \text{ if } t \in T \\ e_{i(t)} \text{ if } t \in \mathcal{I} \setminus T \end{cases}$$

By definition, for every $t \in \mathcal{I} \setminus T$, $e_{i(t)} \notin P_{i(t)}(e_{i(t)})$ and therefore x could be rejected by the coalition T. Therefore, $0 \notin \mathcal{K}$.

Finally, since $\mathcal{K}$ is compact there exists $p \in I\!R^L \setminus \{0_{I\!R^L}\}$ and $\varepsilon > 0$ such that $\varepsilon < \min p \cdot \mathcal{K}$. For every $a \in A$, let $m'_a = p \cdot (x_a - e_a) + \varepsilon/2$ and set $q = 1$. Then, of course for every $t \in \mathcal{I}$, $p \cdot x_t < p \cdot e_t + qm'_i < \min p \cdot P_t(x_t)$, which ends the proof. □

Remark 3.1 *Rationing equilibria without money may be rejected.*
In the equilibrium definition we did not impose that the price of fiat money is positive. The present example shows that a positive price of fiat money is needed in order to ensure that an equilibrium allocation is in the rejective core. Consider an exchange economy with three types of consumers (with $\mathcal{L}(T_1) = \mathcal{L}(T_2) = \mathcal{L}(T_3)$) and two commodities: for all $i \in I$, $X_i = \{0, 1, \ldots, 5\}^2$, $u_1(x) = -x^1 - x^2$, $u_2(x) = -\|x - (1,1)\|_1$, $u_3(x) = -\|x - (0,1)\|_1$, $e_1 = (0,4)$, $e_2 = (0,0)$, $e_3 = (1,0)$. The type symmetric allocation $x_1 = (0,0)$, $x_2 = (1,2)$, $x_3 = (0,2)$ is a rationing equilibrium with $p = q = 0$, $K = \{t(0,-1) \,|\, t \geq 0\}$. However this rationing equilibrium is not in the rejective core since the players of type 2 and 3 may reject with the type symmetric allocation $\xi_1 = (0,2), \xi_2 = (1,1)$ and $\xi_3 = (0,1)$.

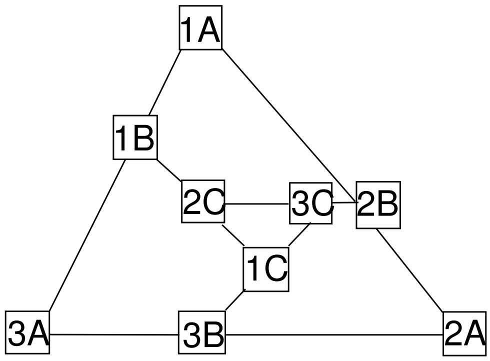
To end this section, we use an example from Shapley and Scarf (1974) to illustrate some facts mentioned in this section.

Example 3.1 Shapley and Scarf (1974) gave the following example in order to show that the core may be empty when commodities are indivisible. We consider an economy with three types of agents $I = \{1, 2, 3\}$ nine commodities $L = \{1_A, 1_B, 1_C, \ldots, 3_C\}$, commodity sets $X_i = \{0, 1\}^9$ and concave utility functions for $i \in I$

$$u_i(x) = \max\{2\min\{x_{i_A}, x_{i+1_A}, x_{i+1_B}\}; \min\{x_{i_C}, x_{i+2_B}, x_{i+2_C}\}\}.$$

The indices are module 3. Initial endowments are $e_i = (e_{ih}) \in X_i$ with $e_{ih} = 1$ if and only if $h \in \{i_A, i_B, i_C\}$.

The following picture illustrates endowments and preferences. Each consumer would like to have three commodities on a straight line containing only one of his commodities. The best bundle is to own a long line containing his commodity i_A and $i + 1_B, i + 1_A$ and the second best would be to own a short line containing his commodity i_C and $i + 2_B, i + 2_C$.



If there is only one agent per type this reduces indeed to Shapley and Scarf's (1974) setting. In this case, at any feasible allocation for some $i \in I$, agent i obtains utility zero and agent $i+2$ at most utility one. However, if they form a coalition it is possible to give utility one to i and two to $i + 2$. Thus, the core is empty.

With an even number of agents per type or a continuum of measure one per type the weak and the rejective core correspond to the allocations such that half of the consumers of type i consume $x_{ih} = 1$ for all $h \in \{i_A, i + 1_A, i + 1_B\}$ and the other half consumes $x_{ih} = 1$ for all $h \in \{i_C, i + 2_B, i + 2_C\}$. So every consumer obtains at least his second best allocation. It is not possible to block an allocation in the sense that all consumers who block are strictly better off. Indeed, they would all need to obtain their best allocation and this is not feasible for any group. To see that this is the only allocation in the core, note that at any other allocation at least one consumer say a consumer of type 1 (or a non-negligible group of a given type) would necessarily get an allocation which yields zero utility. Then by feasibility, a consumer of type 3 (or a non-negligible group of type 3) obtain only their second best choice. The consumer of type 1 can propose the commodities $1_A, 1_B$ in exchange for $3_B, 3_C$ making everybody strictly better off.

Allocations in the core are supported by a uniform distribution of paper money $m_i = m > 0$ for all $i \in I$ and the price $p = (2, 1, 1, 2, 1, 1, 2, 1, 1)$, $q = 1/m$. Thus, a Walras equilibrium with money does not exist for a uniform distribution of paper money. A rationing equilibrium, however, exists. If half of each type obtains one unit of paper money and the other half strictly less than one unit, then the core allocation is a Walras equilibrium allocation with the same price p and $q = 1$.

4 Welfare Analysis

As we mentioned in the introduction, a rationing equilibrium will not necessarily be a strong Pareto optimum[4]. This comes from the fact that in presence of indivisible goods some consumers may own commodities that are worthless to them as a consumption good since the value of these commodities may be so small at the equilibrium that selling them does not enable to buy more of the indivisible goods they are interested.

If the preference relation of at least one consumer is empty valued for all allocations then any feasible allocation would be a weak Pareto optimum. Off course we could have pathologic weak optima even under less extreme circumstances. This motivates the following definition which was also used in Florig (2001).

Definition 4.1 A collection $(x, y) \in A(\mathcal{E})$ is a **Pareto optimum** if there does not exist $(x', y') \in A(\mathcal{E})$ and a non-negligible set $T \subset \mathcal{I}$ such that for a.e. $t \in T$, $x'_t \in P_t(x_t)$ and for a.e. $t \in \mathcal{I}$, $x'_t \neq x_t$ if and only if $t \in T$.

Proposition 4.1 *First Welfare Theorem.*

Every rationing equilibrium is a Pareto optimum.

Proof.

Let (x, y, p, q, K) be a rationing equilibrium and $(x', y') \in A(\mathcal{E})$ Pareto dominating (x, y), with T the non-negligible set from Definition 4.1. From feasibility we already know that

$$e = \int_{\mathcal{I}} x'_t - \int_{\mathcal{J}} y'_t = \int_{\mathcal{I}} x_t - \int_{\mathcal{J}} y_t$$

Therefore,

$$\int_{\mathcal{J}} y'_t - y_t = \int_{\mathcal{I}} x'_t - x_t$$

and since for a.e. $t \in \mathcal{I}$, $x_t \in \delta_t(p, q, K)$ we have for a.e. $t \in \mathcal{I}$ with $x_t \neq x'_t$, $p \cdot (x'_t - x_t) \geq 0$ and thus $p \cdot \int_{\mathcal{I}} x'_t - x_t \geq 0$. By profit maximization $p \cdot \int_{\mathcal{J}} y'_t - y_t \leq 0$. Therefore $p \cdot \int_{\mathcal{I}} x'_t - x_t = p \cdot \int_{\mathcal{J}} y'_t - y_t = 0$ and moreover $p \cdot (x'_t - x_t) = 0$. Therefore for a.e. $t \in \mathcal{I}$ with $x_t \neq x'_t$, $(x'_t - x_t) \in K$ and $\int_{\mathcal{I}} x'_t - x_t \in K$. By the supply definition we have

[4] We recall that a feasible allocation is a strong Pareto optimum if there does not exists another feasible allocation which is preferred to the original one by all and strictly preferred for some consumers. The allocation is a weak Pareto optimum if there does not exists another feasible allocation which is strictly preferred by all.

now for a.e. $t \in \mathcal{J}$, $p \cdot \int_{\mathcal{J}} y'_t - y_t = 0$ and therefore $y'_t - y_t \in -K$ and $\int_{\mathcal{J}} y'_t - y_t \in -K$. Thus

$$K \ni \int_{\mathcal{I}} x'_t - x_t = \int_{\mathcal{J}} y'_t - y_t \in -K$$

which implies $\int_{\mathcal{I}} x'_t - x_t = 0_{I\!R^L}$ and since for a.e. $t \in \mathcal{I}$ with $x_t \neq x'_t$ we have $x'_t - x_t \in K$, and since K is a pointed cone, we have for a.e. $t \in \mathcal{I}$ $x_t = x'_t$ - a contradiction. □

Proposition 4.2 *Second Welfare Theorem.*

Let $\mathcal{E}$ be an economy with $J = \emptyset$ (exchange economy). If x is a Pareto optimum, then there exists $p \in I\!R^L \setminus \{0\}$ and $e' \in X$ such that (x, p) is a Walras equilibrium of $\mathcal{E}'$ which is obtained from $\mathcal{E}$, replacing the initial endowment e by e'.

Proof. For all $t \in \mathcal{I}$ set $e'_{i(t)} = x_{i(t)}$. Since the number of types is finite and the consumption sets are finite, we can define a finite set of consumer types $K \equiv \{1, \ldots, K\}$ satisfying the following

(i) $(T_k)_{k \in K}$ is a finer partition of $\mathcal{I}$ than $(T_i)_{i \in I}$,

(ii) for every $k \in K$, there exists x_k such that for every $t \in T_k$, $x_t = x_k$.

Define

$$H_k = \mathcal{L}(T_k) \left(\mathrm{co} P_k(x_k) - x_k \right)$$

and

$$\mathcal{H} = \mathrm{co} \cup_{k \in K} H_k.$$

First of all, note that $0_{I\!R^L} \notin \mathcal{H}$. Otherwise there exist $(\lambda_k) \in [0,1]^K$ with $\sum_{k \in K} \lambda_k = 1$ and $\xi_k \in \mathrm{co} P_k(x_k)$ for all $k \in K$, such that $\sum_{k \in K} \lambda_k \mathcal{L}(T_k)(\xi_k - x_k) = 0$. Thus there exists $\xi \in X$ such that for all $k \in K$

$$\mathcal{L}(\{t \in T_k \mid \xi_t \in P_{i(t)}(x_t)\}) = \lambda_k \mathcal{L}(T_k)$$

and

$$\mathcal{L}(\{t \in T_k \mid \xi_t = x_t\}) = (1 - \lambda_k) \mathcal{L}(T_k)$$

contradicting Pareto optimality of x.

Due to $\mathcal{H}$ is compact, there exists $p \in I\!R^L \setminus \{0\}$ and $\varepsilon > 0$ such that for all $z \in \mathcal{H}$, $p \cdot z > \varepsilon$. Hence for a.e. $t \in \mathcal{I}$,

$$P_{i(t)}(x_t) \cap \{\xi \in X_{i(t)} \mid p \cdot \xi \leq p \cdot x_t + \varepsilon\} = \emptyset.$$

So (x, p) is indeed a Walras equilibrium of $\mathcal{E}'$. Setting $q > 0$ such that for all $i \in I$, $qm_i < \varepsilon/2$, (x, p, q) would also be a Walras equilibrium with a positive value of paper money, which ends the demonstration of the Second Welfare Theorem. □

Remark 4.1 Under the assumptions of the previous proposition, we could also decentralize any Pareto optimum x by collecting taxes $\tau_t = p \cdot (x_t - e_{i(t)}) + m_t$ from agent $t \in \mathcal{I}$ payable in monetary units. Then, x becomes an equilibrium together with $q = 1$ and p as in the previous proof.

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