The equilibrium manifold with Boundary constraints on the Consumption sets*

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Abstract

In this paper we consider a class of pure exchange economies in which the consumption plans may be restricted to be above a minimal level. This class is parameterised by the initial endowments and the constraints on the consumption. We show that the demand functions are locally Lipschitzian and almost everywhere continuously differentiable even if some constraints may be binding. We then study the equilibrium manifold that is the graph of the correspondence which associates the equilibrium price vectors to the parameters. Using an adapted definition of regularity, we show that: the set of regular economies is open and of full measure; for each regular economy, there exists a finite odd number of equilibria and for each equilibrium price, there exists a local differentiable selection of the equilibrium manifold which selects the given price vector. In the last section, we show that the above results hold true when the constraints are fixed.

Keywords: demand function, general equilibrium, regular economies.

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1.- Introduction

The study of the sensibility of the equilibrium prices with respect to the parameters which define an economy, is a central question in the general equilibrium theory. After a pioneer work of Debreu ([6]), this problem was extensively studied by numerous authors (see for example Balasko ([1]), Mas-Colell ([11]), Smale ([15])).

In this chapter, we investigate the case of a pure exchange economy in which the consumers may face boundary constraints which means that the consumption plans are restricted to be above some minimal positive levels for some commodities. In other words, the admissible consumption plans are the non-negative baskets of commodities greater than a lower bound which may have positive components. Furthermore, the constraints are considered as a parameter together with the initial endowments. Thus, we consider a class of economies in which the consumption sets vary.

The motivation to study this framework comes from applications to imperfect competition models. Indeed, this paper is a first step to analyze the properties of oligopoly equilibria introduced in Codognato-Gabszewicz ([3]) and Gabszewicz-Michel ([10]). A related notion, called Nash equilibrium of a no-destruction Walrasian endowment game, is studied in Safra ([13]). The strategic variables of the agents are the quantities of commodities they put on the market. Taking into account the effect of these quantities on the equilibrium. The fact that the consumers keep part of the initial endowments out of the market, implies that the demand of the consumers depends on these quantities which can be viewed through a simple transformation, as a lower bound on the consumption plan. Obviously, it is essential to know how the equilibrium price on the market depends on the strategies of the agents, to deduce information about the oligopoly equilibria. That is precisely our aim in this paper

In the standard differentiable approach of the economic equilibrium, assumptions are done in order to get differentiable demand function. This has been generalized by several authors in different models and in particular in the exchange economies, to allow the case where the solutions of the optimisation problems are not differentiable everywhere (see, for example, Radner ([12]), Smale([15]), Villanacci ([16,17,18])).

Nevertheless, the technique of the proof in these papers uses a trick to come back to the differentiable case. They consider an extended system of equations to represent the first order necessary and sufficient conditions for the consumer problem by splitting the complementary slackness conditions. This allows to prove the differentiability of the demand functions on open domains by a transversality argument and then, to apply the standard tools of differential topology.

In our model, the utility functions of the agents are kept fixed and the initial endowments and the levels of the boundary constraints are the parameters which define an economy. We posit a standard assumption on the utility functions of the consumers, that is positive gradient vectors, negative definite Hessian matrices on the orthogonal of the gradient vectors and a boundary condition on the positive orthant. The demand functions depend as usual on the prices and the wealth but also on the level of the constraints. They are not everywhere differentiable with respect to these arguments but standard results on mathematical programming imply that they are locally Lipschitzian. Our strategy in the proof is to take advantage of this fact since Rademacher`s theorem implies that the demand functions are almost everywhere differentiable. A simple argument shows that if they are differentiable at a point. Consequently, the set on which the demand functions are not differentiable is a closed null set.

The remainder of the paper is borrowed from Balasko´s works. We consider the equilibrium manifold which is the graph of the correspondence which associates the equilibrium price vectors to an economy. We propose a global parameterization of the equilibrium manifold. From the properties of the demand functions, one deduces that it is lipeomorphic to an open connected subset of an Euclidean space. Through the lipeomorphism, we can define what we called an extended projection which is defined between two open subsets of Euclidean spaces of the same dimension. This mapping is continuously differentiable on an open set which is of full Lebesgue measure and locally Lipschitzian everywhere. We define a singular economy by the fact that it is the image either of a point where the extended projection is not differentiable or of a point where the Jacobian matrix is not onto. Similar definitions are used in different works dealing with nonsmooth mappings. Using the fact that the image of a null set by a locally Lipschitzian mapping is a null set if the dimension of the spaces are the same and Sard's Theorem, we can conclude that the set of the singular economies is a null set.

Then, using standard technics and the fact that the domain of the extended projection is connected, we obtain the same results as in the case without boundary constraints. This just needs the homotopy invariance of the degree since we can compute it at a point where the boundary constraints do not matter.

Thus, for each regular economy, there is a finite odd number of equilibrium prices and for each such price, there exists a local differentiable selection of the equilibrium manifold which selects the chosen price. In other words, an economy with boundary constraints has the same properties as a standard one if we define the regularity as above although the consumption sets change.

In the last section, we consider the case where the boundary constraints are fixed. Using the previous analysis, we show that the same results hold true then the initial endowments are the parameters which define an economy as usual in this literature.

To situate our work with respect to the paper of Shannon ([14]), we study a more specific model but we posit our assumptions on the fundamentals of the economy. Our framework may certainly be generalized by using tools of the nonsmooth analysis but its interest is the fact that we just need standard and well known results to obtain our results.

With respect to ([15,18]), our contribution is essentially on the mathematical method. Nevertheless, the global structure of the equilibrium manifold seems to be a new result in this framework and can not be deduced directly from the fact that the demand functions are almost everywhere differentiable since this does not imply that they are locally Lipschitzian. The gain between continuity and lipschitzianity may seem small but for further applications, this allows to use the tools coming groom nosmooth analysis. Furthermore, we do not need to assume as in ([18]) that there exists an interior Pareto optimal allocation. This assumption is not stated in terms of the fundamentals of the economy it is not easy to check it. We can remove this hypothesis since we incorporate the levels of the constraints in the parameters to define an economy. From a mathematical point of view, our argument seems to be simpler and shorter since we extensively use previous results. The Lipschitzianity of the demands allows us to remove a transversality argument. We can expect further development by considering more general constraints or abstract set constraints which will be hardly managed by the differentiable technics.

This chapter is organized as follows: in Section 2, we describe the model, posit the assumptions and study the properties of the demand function. In Section 3, we define the parameterization of the equilibrium manifold and we study the extended projection which leads to the main result. Finally, Section 4 deals with the case of fixes constraints.

2.- The model and the properties of the demand function

We consider a class of exchange economies with positive numbers ℓ of commodities and m of consumers. Let $L := \{1,...,\ell\}$AND $M := \{1,...,m\}$[1].

Given an agent $i \in M$, we assume the existence of an initial restriction for her / his consumption which is represented by a vector $\xi_i \in R^\ell$. This means that the *consumption set* of the consumer is

$$\mathrm{X}_i(\xi_i) := \{\chi \in R^\ell_{++} | \xi_i \le \chi\}.$$

Her/his preferences are represented by the restriction of a utility function

$$u_i : R^\ell_{++} \to R$$

[1] In this paper we use the following notation: if $\chi = (\chi_j)$ and $y = (y_j)$ are vectors of $R^n, \chi \le (\langle\langle)y$ means $\chi_j \le (\langle)y_j$ for each $j = 1,...,n.$ Note that we use $\le$ also for real numbers. It should be clear in the context. We will consider the sets $R^n_+ = \{\chi \in R^n \ |0 \le \chi\}$ and $R^n_{++} = \{\chi \in R^n \ |0\langle\langle \chi\}\cdot \chi \cdot y = \sum_{j=1}^n \chi_j y_j$ denotes the inner product of χ and y.

to her/his consumption set. We posit the following assumption on the utility functions.

Assumption C. For each $i \in M, u_i$ is a C^2 mapping, for each $\chi \in R^{\ell}_{++,} \nabla u_i(\chi) \notin -R^{\ell}_{+}$, for all sequence (χ^{ν}) of R^{ℓ}_{++} which converges to $\chi \in \partial R^{\ell}_{++}$, $\left(\frac{1}{\|\nabla u_i(\chi^{\nu})\|} \nabla u_i(\chi^{\nu}) \cdot \chi^{\nu} \right)$ converges to O, and $D^2 u_i(\chi)$ is negative definite on $\nabla u_i(\chi)^{\perp 2}$.

Note that Assumption C is weaker than the usual conditions. In particular, we do not assume that the preferences are strictly monotone. Furthermore, the indifference curves may cross the boundary. This is the case, for example, for the utility function $u_i(\chi) = \sum_{h \in L} \sqrt{\chi_h}$ which satisfies Assumption C but not the assumptions of the following lemma. The proof is given in Appendix.

Lemma 2.1 *If the utility function u_i is a C^2 mapping such that, for each $\chi \in R^{\ell}_{++}, \nabla u_i(\chi) \in R^{\ell}_{++}, \{\chi' \in R^{\ell}_{++} \mid u_i(\chi) \leq u_i(\chi')\}$ is a closed subset of R^{ℓ} and $D^2 u_i(\chi)$ is negative definitive on $\nabla u_i(\chi)^{\perp}$, then it satisfies Assumption P.*

Finally, we assume that each consumer has an initial endowment of commodities denoted $\omega_i \in R^{\ell}_{+}$ that allows him /her to participate to the exchange. In order to deal with endowments on the boundary of the consumption set, we denote by $L_i \subset L$, the set of commodities that the consumer i can obtain as initial endowment. This means that $\omega_{ih} \rangle 0$ if $h \in L_i$ and $\omega_{ih} = 0$ if $h \notin L_i$. We assume that each commodity is available on the market, which means that the total initial endowment $\omega = \sum_{i \in M} \omega_i \in R^{\ell}_{++}$ or, in other words, $\cup_{i \in M} L_i = L$. In the following, $R^{Li} = \{\chi \in R^{\ell} \mid \chi_h = 0, \forall h \notin L_i\}$.

[2] Orthogonal complement of the vector $\nabla u_i(\chi)$

We normalize the price vectors by considering $S := \{p \in R^{\ell}_{++} \; | p_{\ell} = 1\}$ as the space of prices.

In the following, the utility functions will remain fixed. Consequently, we define and *economy* as a point $(\omega_i, \xi_i)_{i \in M} \in \Pi_{i \in M} (R^{Li}_{++})^2$, such that for each $i \in M$, for each $h \in L_i, \xi_{ih} \langle \omega_{ih}$. The set of economies will be denoted by E.

The demand of the ith consumer is the set of solutions of the following optimisation problem:

$$\begin{cases} \max u_i(\chi) \\ p \cdot \chi \leq r_i \\ \chi \in \mathrm{X}_i(\xi_i) \end{cases}$$

where $p \in S$ is a given price vector and $r_i \in R$ is the *wealth* of the consumer.

From Assumption C, one easily deduces that this problem has a unique solution when $p \cdot \xi_i^+ \langle r_i$ where ξ_i^+ is the projection of ξ_i on R^{ℓ}_+. This mapping is called the *demand function* of the ith consumer and is denoted by $f_i(p, r_i, \xi_i)$.

The remaining of this section is devoted to the properties of the mappings f_i since they play a crucial role in the analysis of the equilibrium manifold. To prepare the case where the constraints parameters (ξ_i) are fixed, we also look at the partial differentiability with respect to (p, r_i).

Let Ω be the open subset of $S \times R \times R^{\ell}$ defined by

$$\Omega := \{(p, r_i, \xi_i) \in S \times R_{++} \times R^{\ell} \; | p \cdot \xi_i^+ \langle r_i\}.$$

And for each $\xi_i \in R^\ell$, let

$$\Omega^{\mathrm{E}i} := \{(p, r_i) \in S \times R_{++} \ |(p, r_i, \xi_i) \in \Omega\}$$

The continuity of f_i is a direct consequence of the maximum theorem $(Berge[2])$ and the nonsatiation of the utility functions implies that f_i satisfies the Walras law, that is $p \cdot f_i(p, r_i, \xi_i) = r_i$. The main goal of this section is to prove the following proposition.

Proposition 2.1. *For each $i \in M$, the demand function $f_i : \Omega \to R^\ell_{++}$ is locally Lipschitzian. There exists an open subset Ω_i of Ω such that $\Omega \setminus \Omega_i$ is a null set with respect to the Lebesgue measure and f_i is continuously differentiable on Ω_i. For each $\xi_i \in R^\ell, \Omega_i^{\xi i} = \{(p, r_i) \in S \times R_{++} |(p, r_i, \xi_i) \in \Omega_i\}$ is an open subset of $\Omega^{\xi i}$ such that $\Omega^{\xi i} \setminus \Omega_i^{\xi i}$ is a null set.*

Note that for each $\xi_i \in R^\ell, f_i(.,.,\xi_i)$ is continuously differentiable on $\Omega_i^{\xi i}$. We prepare the proof of this proposition by a lemma. Let us first recall that the first order necessary conditions at $\chi \in R^\ell_{++}$ for the consumer problem are: there exist $\lambda \geq 0$ and $u \in R^\ell_+$ such that

$$\begin{cases} \nabla u_i(\chi) = \lambda p - u \\ u \cdot (\chi - \xi_i) = 0 \\ p \cdot \chi = r_i, \chi \geq \xi_i \end{cases}$$

The strict complementarity slackness condition holds if $x_h = \xi_h$ implies $u_h \rangle 0$ for all h. If this condition holds in a neighborhood of (p, r_i, ξ_i) and f_i is C^1 on this neighborhood (See Fiacco-McCormick [9]).

Lemma 2.2 If the strict complementarity slackness condition does not hold for the with consumer problem then the demand function is not differentiable at (p, r_i, ξ_i) with respect to (p, r_i), thus also with respect to (p, r_i, ξ_i).

Proof. Let $(p, r_i, \xi_i) \in \Omega$ and let $\overline{\chi} = f_i(p, r_i, \xi_i)$. Then, there exists $\lambda \rangle 0$ and $u \in R_+^\ell$ such that

$$\begin{cases} \nabla u_i(\overline{\chi}) = \lambda \cdot p - u \\ u \cdot (\overline{\chi} - \xi_i) = 0 \\ p \cdot \overline{\chi} = r_i, \overline{\chi} \geq \xi_i \end{cases}$$

If the strict complementarity slackness condition does not hold at (p, r_i, ξ_i) then $L_0 = \{h \in L \ | f_{ih}(p, r_i, \xi_i) = \xi_{ih} and \quad u_h = 0\}$ is nonempty. Let, for all *t* in a neighborhood of 0 in $R, \chi(t) = f_i(p, r_i, \xi_i) + t\sum_{h \in L_0} e^h$ where e^h is the hth vector of the canonical basis of R^ℓ. Let

$$p(t) = \frac{1}{\lambda}(\nabla u_i(t)) + u) \text{ and } r_i(t) = p(t) \cdot \chi(t).$$

For $t \rangle 0, \chi(t) = f_i(p(t), r_i(t), \xi_i)$ since the first order necessary conditions are satisfied with λ and u. Consequently, if f_i is differentiable at (p, r_i, ξ_i) with respect to (p, r_i) for all $h \in L_0, f_{ih}(p(t), r_i(t)\xi_i)$ is differentiable with respect to *t* and the derivative at $t = 0$ is equal to 1. But this implies that this function is strictly increasing in a neighborhood of 0 which leads to a contradiction since $f_{ih}(p(0), r_i(0), \xi_i) = \xi_{ih}$ and $f_{ih}(p(t), r_i(t), \xi_i) \geq \xi_{ih}$ for every *t*. This prove that f_i is not differentiable at (p, r_i, ξ_i) with respect to (p, r_i)

Proof of Proposition 2.1. To prove that f_i is locally Lipschitizian, we first remark that $f_i(p,r_i,\xi_i) \neq \xi_i$ due to the Walras law and $p \cdot \xi_i \langle r_i$. Thus the active constraints at $f_i(p,r_i,\xi_i)$ are linearly independent. This remark together with the fact that $D^2 u_i(\chi)$ is definite negative on $\nabla u_i(\chi)^{\perp}$, allows us to apply Cornet-Laroque $[4]$ or Cornet-Vial $[5]$ which leads to the results.

Consequently, we know that the demand function is almost everywhere differentiable since it is locally Lipschitzian (Rademacher´s Theorem) and this is also true for $f_i(.,.,\xi_i)$ for each ξ_i. Lemma 2.1. shows that if $f_i(.,.,\xi_i)$ is differentiable, then the strict complementarity slackness condition hold true and then f_i and $f_i(.,.,\xi_i)$ are continuously differentiable in a neighborhood. The result is a direct consequence of this remark if we define Ω_ι as the set on which f_i is differentiable since $\Omega_i^{\xi i}$ is then the set on which $f_i(.,.,\xi_i)$ is differentiable.

3.- The equilibrium Manifold

In this section, we study the equilibrium price vectors associated with an economy $((\omega_i),(\xi_i))$ from a global point of view as in Balasko $([1])$. A price vector $p \in S$ is an equilibrium price for the economy $((\omega_i),(\xi_i)) \in \mathrm{E}$ if the total demand at $p \in S$ is equal to the supply, that is,

$$\sum_{i=1}^{m} f_i(p, p \cdot \omega_i, \xi_i) = \sum_{i=1}^{m} \omega_i$$

In that case we shall say that $(p,(\omega_i),(\xi_i)) \in S \times E$ is an *equilibrium point* and the *equilibrium manifold* $\mathrm{E}_{eq} \subseteq S \times \mathrm{E}$ is defined as the set of equilibrium points in $S \times \mathrm{E}$.

In the framework of this paper, the equilibrium manifold is not necessarily differentiable, and then we are unable to apply directly standard results of differential topology to study the natural projection, that is, the mapping $\pi : \mathrm{E}_{eq} \to \mathrm{E}$ such that

$$\pi(p,(\omega_i),(\xi_i)) = ((\omega_i),(\xi_i)).$$

Our approach consists in defining a suitable parameterization of the equilibrium manifold and then to work on an open subset of an Euclidean space to obtain the desired results. We recall that we are working with locally Lipschitzian and a.e. continuously differentiable mapping, whereas in the standard case, they are smooth everywhere.

In the following, to simplify the exposition, if χ is a vector of R^{ℓ}, then $\overline{\chi}$ is the vector of $R^{\ell-1}$ obtained by suppressing the last coordinate of χ. We now define an open subset U of an Euclidean space and we then prove that this set is connected and locally lipeormorphic to the equilibrium manifold. Let $U \subseteq S \times R^{m}_{++} \times R^{(\ell-1)(m-1)}_{++} \times R^{\ell m}$ be the subset defined as follows: $\left(p,(r_i),\left(\overline{\omega}_i\right)_{i=1}^{m-1},(\xi_i)\right) \in U$ if

- for each $i = 1,\dots,m-1, \overline{\xi}_i \langle\langle \overline{\omega}_i$,
- for each $i = 1,\dots,m-1, \max\{\xi_{i\ell},0\} + \sum_{h=1}^{\ell-1} p_h \overline{\omega}_{ih} \langle r_i$;
- max $\{0,\xi_m\} + \sum_{i=1}^{m-1} \omega_i \langle\langle \sum_{i=1}^{m} f_i(p,r_i,\xi_i)$,

  where $\omega_i := \left(\overline{\omega}_i, r_i - \sum_{h=1}^{\ell-1} p_h \overline{\omega}_{ih}\right), i = 1,\dots,m-1$

Proposition 3.1. U is an open connected subset of $S \times R^{m}_{++} \times R^{(\ell-1)(m-1)}_{++} \times R^{\ell m}$.

Proof. The openness is a direct consequence of the definition. Now, in order to prove the connectedness of U, let $\mu^* = \left(p^*, (r_i^*), (\overline{\omega}_i^*)_{i=1}^{m-1}, (\xi_i^*) \right)$ be an element of U. The first part of the proof shows that this point is connected to the point $\overline{\mu} = \left(p^*, (r_i^*), (\overline{f}_i(p^*, r_i^*, 0))_{i=1}^{m-1}, (0) \right)$.

From the definition of U, one has $\overline{\xi}_i^* \langle\langle \overline{\omega}_i^*$ for each $i = 1, ..., m-1$ and $\max\{0, \xi_m^*\} + \sum_{i=1}^{m-1} \omega_i^* \langle\langle \sum_{i=1}^{m} f_i(p^*, r_i^*, \xi_i^*)$. Since $\xi_m^* \leq \max\{0, \xi_m^*\}$, one deduces $\sum_{i=1}^{m} \xi_i^* \langle\langle \sum_{i=1}^{m} f_i(p^*, r_i^*, \xi_i^*)$. Consequently, for all $h \in L$, there exists $i^h \in M$ such that $\xi_{i^h h}^* \langle f_{i^h h}(p^*, r_{i^h}^*, \xi_{i^h}^*)$. For all $i \in M$, let $\mathrm{H}_i^* = \{ h \in L \mid \xi_{ih}^* \langle f_{ih}(p^*, r_i^*, \xi_i^*) \}$. From the properties of f_i, one remarks that for all $t \in [\{0,1\}]$:

$$f_i(p^*, r_i^*, \xi_i^*) = f_i\left(p^*, r_i^*, \xi_i^* - t \sum_{h \in H_i^*} \xi_{ih}^* e^h \right).$$

Hence, the path $\left(p^*, (r_i^*), (\overline{\omega}_i^*)_{i=1}^{m-1}, (\xi_i^t) \right)$, where $\xi_i^t = \xi_i^* - t \sum_{h \in H_i^*} \xi_{ih}^* e^h$, remains in U. Thus μ^* is connected to $\mu^1 = \left(p^*, (r_i^*), (\overline{\omega}_i^*)_{i=1}^{m-1}, (\tilde{\xi}_i) \right)$ where $\tilde{\xi}_i = \xi_i^0$. We point out that for each $h, \tilde{\xi}_{i^h h} = 0$. Thus, for all $t \in [0,1]$,

$$0 = t\tilde{\xi}_{i^h h} \langle f_{i^h h}(p^*, r_{i^h}^*, t\tilde{\xi}_{i^h})$$

Recalling that, for all $i \in M, t\tilde{\xi}_i \leq f_i(p^*, r_i^*, t\tilde{\xi}_i)$, one deduces that

$$t\sum_{i=1}^{m} \tilde{\xi}_i \langle\langle \sum_{i=1}^{m} f_i(p^*, r_i^*, t\tilde{\xi}_i).$$

In the following, for $t \in [0,1]$, σ^t denotes the strictly positive vector

$$\sum_{i=1}^{m} f_i\left(p^*, r_i^*, t\widetilde{\xi}_i\right) - t\sum_{i=1}^{m} \widetilde{\xi}_i$$

and

$$\omega_i^t = t\widetilde{\xi}_i + \frac{r_i^* - p^* \cdot t\widetilde{\xi}_i}{p^* \cdot \sigma^t}\sigma^t .$$

Note that $\sum_{i=1}^{m} \frac{r_i^* - p^* \cdot t\widetilde{\xi}_i}{p^* \cdot \sigma^t} = 1$ since $p^* \cdot f_i\left(p^*, r_i^*, t\widetilde{\xi}_i\right) = r_i^*$ for each i. One easily checks that $p^* \cdot \omega_i^t = r_i^*, t\widetilde{\xi} \langle\langle \omega_i^t$ for every $i \in M$ and $\sum_{i=1}^{m} \omega_i^t = \sum_{i=1}^{m} f_i\left(p^*, r_i^*, t\widetilde{\xi}_i\right)$. Hence, $\left(p^*, \left(r_i^*\right), \left(\overline{\omega}_i^t\right) i = 1,..., m-1, \left(t\widetilde{\xi}_i\right)\right)$ is an element of U.

The points μ^1 and $\mu^2 = \left(p^*, \left(r_i^*\right), \left(\overline{\omega}_i^1\right)_{i=1}^{m-1}, \left(\widetilde{\xi}_i\right)\right)$ are connected by a simple convex combination.

The points μ^2 and $\mu^3 = \left(p^*, \left(r_i^*\right), \left(\overline{\omega}_i^0\right)_{i=1}^{m-1}, (0)\right)$ are connected by the path $\left(\left(p^*, \left(r_i^*\right), \left(\overline{\omega}_i^t\right)_{i=1}^{m-1}, \left(t\widetilde{\xi}_i\right)\right)\right)_{t\in[0,1]}$.

Finally, the points μ^3 and $\overline{\mu}$ are connected by a convex combination.

To end the proof, we just remark that for all $\left(p', \left(r_i'\right)\right) \in S \times R_{++}^m$, the points $\overline{\mu}$ and $\left(p', \left(r_i'\right), \left(\overline{f}_i\left(p', r_i', 0\right)_{i=1}^{m-1}, (0)\right)\right)$ are connected by the path

$$\left(tp^* + (1-t)p', \left(tr_i^* + (1-t)r_i'\right), \left(\overline{f}_i\left(tp^* + (1-t)p', tr_i^* + (1-t)r_i', 0\right)_{i=1}^{m-1}, (0)\right)\right)_{t\in[0,1]}$$

We now define an open subset V of U which plays an important role in what follows since the mapping θ, which allows us to parameterize the equilibrium manifold, is differentiable on V. Thus, $\left(p,(r_i),(\overline{\omega}_i)_{i=1}^{m-1},(\xi_i)\right)$ belongs to V if, for each $i,(p,r_i,\xi_i)$ belongs to Ω_i which is given by Proposition 2.1. Of $\left(p,(r_i),(\overline{\omega}_i)_{i=1}^{m-1},(\xi_i)\right)$ belongs to V, the mappings f_i are differentiable at (p,r_i,ξ_i) for each i. Since $\Omega\backslash\Omega_i$ is a closed null set for each I, then $U\backslash V$ is also a closed null set. Let us now define the mappings $\theta: U\to \mathrm{E}_{eq}$ and $\phi:\mathrm{E}_{eq}\to U$ as follows:

$$\theta\left(p,(r_i),(\overline{\omega}_i)_{i=1}^{m-1},(\xi_i)\right):=\left(p,(\omega_i)_{i=1}^{m-1},\sum_{i=1}^{m} f_i(p,r_i,\xi_i)-\sum_{i=1}^{m-1}\omega_i,(\xi_i)\right)$$

with $\omega_i=\left(\overline{\omega}_i,r_i-\sum_{h=1}^{\ell-1}p_h\overline{\omega}_{ih}\right)$ for $i=1,\dots,m-1$

$$\phi(p,(\omega_i),(\xi_i)):=\left(p,(p\cdot\omega_i)_{i=1}^{m},(\overline{\omega}_i)_{i=1}^{m-1},(\xi_i)\right),$$

The definition of $\theta: U\to \mathrm{E}_{eq}$ and $\phi:\mathrm{E}_{eq}\to U$ are directly borrowed from Balasko ([1]) and extended to take into account the parameter ξ.

From the properties of the demand functions (Proposition 2.1.), it is easy to check that both ϕ and θ are locally Lipschitzian mappings and, moreover, θ is continuosly differentiable on V. Besides, it is easy to check that those functions are one to one and onto and by computing $\theta \circ \phi$ and $\phi \circ \theta$ in theirs respective domains, it follows directly that θ is the inverse of ϕ.

Thus, we conclude that E_{eq} is Lipeomorphic[3] to U and these results are summarized in the following proposition.

[3] Two sets are lipeomorphic if it exist a one to one, onto and locally Lipschitzian mapping from the first to the second set with a locally Lipschtizian inverse.

Proposition 3.2. *(i)* θ *and* ϕ *are one to one and onto, and moreover* $\theta^{-1} = \phi$*;*

(ii) θ *and* ϕ *are locally Lipschitzian mappings;*

(iii) θ *is continuously differentiable on V;*

(iv) E_{eq} *is Lipeomorphic to* U.

Thus, as we had mentioned before, a direct consequence of the previous proposition is that E_{eq} may not be a differentiable sub-manifold of $S \times \left(R^{\ell}\right)^m \times \left(R^{\ell}\right)^m$ of θ is not differentiable everywhere and then, we cannot apply the usual properties of differential topology to study it.

In particular, the natural projection $\pi : \mathrm{E}_{eq} \to \mathrm{E}$ is not differentiable and consequently the arguments used in the standard case (see Balasko $[1]$) are not suitable in this nonsmooth setting. This situation enforced us to introduce a "new kind" of projection, namely the *extended projection* $\Pi : U \to \mathrm{E}$ defined as:

$$\Pi := \pi o \theta,$$

that is,

$$\Pi\left(p, (r_i), \left(\overline{\omega}_i\right)_{i=1}^{m-1}, (\xi_i)\right) = \left(\left(\omega_i\right)_{i=1}^{m-1}, \sum_{i=1}^{m} f_1(p, r_i, \xi_i) - \sum_{i=1}^{m-1} \omega_i, (\xi_i)\right).$$

With $\omega_i = \left(\overline{\omega}_i, r_i - \sum_{h=1}^{\ell-1} p_h \overline{\omega}_{ih} \right)$ for $i = 1, \ldots, m-1$

Thus, we have the following proposition.

Proposition 3.3. *The extended projection* $\Pi : U \to \mathrm{E}$ *is a proper, locally Lipschitzian mapping and it is continuously differentiable on V.*

Proof. Except for the properness, the properties of Π are direct consequences of the properties of θ. From the definition of Π, it suffices to show that π is proper since θ is a homeomorphism. Let K be a compact subset of E and let $\left(p^n,\left(\omega_i^n\right),\left(\xi_i^n\right)\right)_{n\geq 1}$ sequence of $\pi^{-1}(\mathrm{K}) \subset \mathrm{E}_{eq}$. Let $\left(\chi_i^n = f_i\left(p^n,\left(\omega_i^n\right),\left(\xi_i^n\right)\right)\right)$. Let $q^n = \frac{1}{\sum_{h=1}^{\ell} p_h^n} p^n$. Since the sequence $\left(q^n\right)$ remains in the simplex of R^ℓ and $\left(\chi_i^n\right)$ is an attainable allocation, it follows that the sequence $\left(q^n,\left(\omega_i^n\right),\left(\xi_i^n\right),\left(\chi_i^n\right)\right)$ remains in a compact set. Thus it has a converging sub-sequence and we denote its limit by $\left(\widetilde{q},\left(\widetilde{\omega}_i\right),\left(\widetilde{\xi}_i\right),\left(\widetilde{\chi}_i\right)\right)$.

To end the proof, we have to show that $\left(\widetilde{q},\left(\widetilde{\chi}_i\right)\right)$ is an equilibrium of the economy $\left(\left(\widetilde{\omega}_i\right),\left(\widetilde{\xi}_i\right)\right)$. Indeed, the strict monotonicity of the preferences implies that each equilibrium price vector is strictly positive hence the subsequence of $\left(p^n\right)$ converges to $\frac{1}{\overline{q}e}\widetilde{q}$ which proves that each sequence of $\pi^{-1}(\mathrm{K})$ has a converging subsequence hence $\pi^{-1}(\mathrm{K})$ is compact.

One obviously has $\widetilde{\xi}_i \leq \widetilde{\chi}_i, \widetilde{q}\cdot\widetilde{\chi}_i \leq \widetilde{q}\cdot\widetilde{\omega}_i$ for all *I* and $\sum_{i=1}^m \widetilde{\chi}_i = \sum_{i=1}^m \widetilde{\omega}_i$. It remains to show that $\widetilde{\chi}_i \in f_i\left(\widetilde{q},\widetilde{q}\cdot\widetilde{\omega}_i,\widetilde{\xi}_i\right)$ for all *i*. If it is not true, there exists *i* and χ_i such that $\widetilde{\xi}_i \leq \chi_i, \widetilde{q}\cdot\chi_i \leq \widetilde{q}\cdot\widetilde{\omega}_i$ and $\mu_i(\chi_i) \rangle \mu_i(\widetilde{\chi}_i)$. By the continuity of μ_i, there exists $\lambda\langle 1$ close to 1 and $\in\rangle 0$ small enough such that $\chi_i' = \lambda\chi_i + (1-\lambda)\widetilde{\xi}_i + \in \sum_{h=1}^{\ell} e^h$ satisfies $\widetilde{\xi}_i \langle\langle \chi_i', \widetilde{q}\cdot\chi_i' \langle \widetilde{q}\cdot\widetilde{\omega}_i$ and $\mu_i(\chi_i')\rangle\mu_i(\widetilde{\chi}_i)$. For η large enough, one has $\xi_i^n \langle\langle \chi_i', q^n\cdot\chi_i' \langle q^n\cdot\omega_i^n$ and $\mu_i(\chi_i')\rangle\mu_i\left(\chi_i^n\right)$ but this contradicts the fact that $\chi_i^n = f_i(q^n, q^n\cdot\omega_i^n, \xi_i^n$. Hence $\widetilde{\chi}_i \in f_i\left(\widetilde{q},\widetilde{q}\cdot\widetilde{\omega}_i,\widetilde{\xi}_i\right)$.

We now come to the definition of a regular (resp. singular) economy. Since Π is locally Lipschitzian, we deduce from $[8]$ that $\Pi\,(\mathrm{U}\backslash V)$ is a null set. Since Π is proper, $\Pi\,(\mathrm{U}\backslash V)$ is closed. Thus, we have the following proposition.

Proposition 3.4. $\Pi\,\mathbf{(U\backslash}\boldsymbol{V}\mathbf{)}$ is a closed null set.

As usual in differential topology, we say that $\chi \in V$ is a critical point of $\Pi\,\big|_V$ if the differential $\partial\Pi\,\big|_V(\chi)$ of $\Pi\,\big|_V$ at χ is not onto. A critical value of $\Pi\,\big|_V$ is an image of a critical point. This leads to the following definition which allows to encompass the fact that the extended projection is not everywhere differentiable. Similar options of regular (resp. singular) value are used in the literature dealing with non-smooth mappings.

Definition 3.1. *An economy* $((\omega_i),(\xi_i)) \in \mathrm{E}$ *will be called regular if it does not belong to* $\Pi(U \setminus V)$ *and it is not the image of a critical point of* $\Pi\,\big|_V$. *An economy is singular if it is not regular. We will denote the set of singular (resp. regular) economies as* E^s *(res.* E^r*)*.

From this last property and the fact that V and E have the same dimension, we are able to apply Sard's Theorem to conclude that the critical values of $\Pi\,\big|_V$ is a null set in E. Furthermore, one easily deduces from the properness of Π and the definition of a singular economy that E^s is closed. Consequently, we can summarize the results on E^s as follows:

Proposition 3.5. E^s *is a closed null set in* E

Using the standard results of differential topology and in particular the Implicit Function Theorem as in the standard case without boundary constraints, we obtain the following results on the equilibrium manifold.

Theorem 3.1. (i) For all $((\omega_i),(\xi_i)) \in \mathrm{E}^r$, there exists a finite number of equilibrium prices.

(ii) *Let* $((\omega_i),(\xi_i)) \in \mathrm{E}^r$ *and* $p \in S$ *be an equilibrium price. Then, there exists a neighborhood* N *of* $((\omega_i),(\xi_i))$, *a neighborhood* N' *of* $p \in S$ *and a differentiable mapping* $q : \mathrm{N} \to \mathrm{N}'$ *such that*

(a) $q(\omega_i),(\xi_i)\ ) = p$,

(b) $q((\omega_i'),(\xi_i'))$ *is the unique equilibrium price of* $((\omega_i'),(\xi_i'))$ *in* N' *for all* $((\omega_i'),(\xi_i')) \in \mathrm{N}$

We end our work by computing the degree of the extended projection Π. Since this mapping is not everywhere continuously differentiable, we need the definition of the degree for a continuous mapping, a concept which is presented for example in Deimling $([7])$. Due to the connectedness of U, it suffices to compute the degree for one value that is for one economy $((\omega_i),(\xi_i))$. Furthermore, if this economy is regular, the degree can be computed by the standard formula for differentiable mapping that is the sum over the element of the inverse image of the sign of the determinant of the Jacobian matrix.

That is why we choose to compute the degree at a point $((\omega_i),(0))$ where (ω_i) is a Pareto optimum of the exchange economy $(\mu_i,\omega_i)_{i=1}^m$ The particularity of such point is first the fact that the boundary constraints do not matter. Indeed, the vectors (ξ_i) remains in a small neighborhood of 0, then the demand functions are the same with or without boundary constraints. Consequently, since in the mapping Π, the parameter (ξ_i) appears only in the last component and it is the identity, the determinant of the Jacobian matrix of Π at a point $\left(p,(r_i),(\overline{\omega_i})_{i=1}^{m-1},(0)\right)$ is the same as the determinant of the mapping Π defined by:

$$\widetilde{\Pi}\left(p,(r_i),(\overline{\omega_i})_{i=1}^{m-1}\right) = \left(\left(\overline{\omega_i}, r_i - \sum_{h=1}^{\ell-1} p_h \overline{\omega_{ih}}\right)_{i=1}^{m-1}, \sum_{i=1}^{m} f_i(p,r_i,0) - \sum_{i=1}^{m-1} \omega_i\right).$$

This mapping is exactly the natural projection studied in Balasko composed by the local diffeomorphism θ.

Taken into account the above remark, the second interest of considering a Pareto optimal initial endowment comes from the fact that there exists a unique equilibrium price vector and the determinant of the Jacobian matrix of the natural projection for this price is not equal to zero as it is proved in Balasko $([1])$. Consequently, the degree of Π is equal to 1 and we have to following results.

Theorem 3.2. *(i)* Π *is of degree 1 and the onto.*

(ii) For all $((\omega_i),(\xi_i)) \in \mathrm{E}^r$ *, there exist a finite odd number of equilibrium prices.*

Note that the above results implies in particular that there exists at least one equilibrium price vector for each economy $((\omega_i),(\xi_i)) \in \mathrm{E}$.

4.- The case of fixed constraints

In this section, we use the previous analysis to study the case where the boundary constraints represented by the parameters $(\tilde{\xi}_i)$ are fixed. Then the economy depends only on the initial endowments which lie on the set

$$\mathrm{E}^{\tilde{\xi}} = \left\{(\omega_i) \in \left(R^{\ell}_{++}\right)^m \;\middle|\; \forall i \in \mathrm{I}, \tilde{\xi}_i \langle\langle \omega_i \right\}.$$

We can then define the sets $\mathrm{E}^{\tilde{\xi}}_{eq}, \mathrm{E}^{\tilde{\xi}r}, U^{\tilde{\xi}}, V^{\tilde{\xi}}$ and the mappings $\theta^{\tilde{\xi}}, \phi^{\tilde{\xi}}$ and $\Pi^{\tilde{\xi}}$ merely by considering the parameter $\tilde{\xi}$ as fixed. Note that the definition of $\Omega_i^{\tilde{\xi}_i}$ in Proposition 2.1. implies that $\left(p,(r_i),(\overline{\omega}_i)_{i=1}^{m-1},(\tilde{\xi}_i)\right)$ belongs to V if $\left(p,(r_i),(\overline{\omega}_i)_{i=1}^{m-1}\right)$ belongs to $V^{\tilde{\xi}}$. All the results given in Proposition 3.1. to Theorem 3.1. still hold except the connectedness of $U^{\tilde{\xi}}$.

Nevertheless, the argument uses to compute the degree of Π does not work since we consider a Pareto optimal initial endowment without constraint. If there is some constraints, the Pareto optimal allocation may be on the boundary of the consumption set which does not allow to choose it as initial endowments. In the following proposition, we use the particular structure of Π and $\Pi^{\tilde{\xi}}$ to prove that the degree is the same for both mappings. Consequently, we obtain exactly the same results as in the standard case without boundary constraints. In other words, the fact that the indifference curves may cross the boundary of the consumption set does not affect the global analysis of the equilibrium manifold except on a null set.

Proposition 4.1. *For each* $(\tilde{\xi}_i) \in (R^{\ell})^m$,

(i) $\Pi^{\tilde{\xi}}$ *is of degree 1 and then onto.*

(ii) *For all* $((\omega_i)) \in \mathrm{E}^{\tilde{\xi}r}$ *, there exists a finite odd number of equilibrium prices.*

Proof. Let $(\tilde{\xi}_i) \in (R^{\ell})^m$ and let $(\omega_i) \in \mathrm{E}^{\tilde{\xi}r}$. Let $\mathrm{P}(\omega_i)$ the finite set of equilibrium price vectors in S associated to (ω_i). Note that from the definition of an equilibrium, $\mathrm{P}(\omega_i)$ is also the finite set of equilibrium price vector in S associated to $((\omega_i),[\tilde{\xi}_i])$. For all $p \in \mathrm{P}(\omega_i)$, let $\left(p,(r_i),(\overline{\omega}_i)_{i=1}^{m-1}\right) \in U^{\tilde{\xi}}$ be the image of $(p,(\omega_i))$ by the mapping $\phi^{\tilde{\xi}}$. Note that $\left(p,(r_i),(\overline{\omega}_i)_{i=1}^{m-1},(\tilde{\xi})\right) \in U$ is the image of $(p,(\omega_i),(\tilde{\xi}_i))$ by the mapping ϕ.

From the definition of a regular economy, $\left(p,(r_i),(\overline{\omega}_i)_{i=1}^{m-1}\right)$ belongs to $V^{\tilde{\xi}}$ for each $p \in \mathrm{P}(\omega_i)$, hence $\left(p,(r_i),(\overline{\omega}_i)_{i=1}^{m-1},(\tilde{\xi}_i)\right)$ belongs to *V*. This implies that Π is differentiable in a neighborhood of $\left(p,(r_i),(\overline{\omega}_i)_{i=1}^{m-1},(\tilde{\xi}_i)\right)$.

Note that

$$\Pi^{\tilde{\xi}}\left(p,(r_i),(\overline{\omega}_i)_{i=1}^{m-1}\right)=\left((\omega_i)_{i=1}^{m-1},\sum_{i=1}^{m} f_i\left(p,r_i,\tilde{\xi}_i\right)-\sum_{i=1}^{m-1}\omega_i\right)$$

and

$$\Pi\left(p,(r_i),\left(\overline{\omega}_i\right)_{i=1}^{m-1},\left(\tilde{\xi}_i\right)\right)=\left(\left(\omega_i\right)_{i=1}^{m-1},\sum_{i=1}^{m} f_i\left(p,r_i,\tilde{\xi}_i\right)-\sum_{i=1}^{m-1}\omega_i,\left(\tilde{\xi}_i\right)\right)$$

with $\omega_i = \left(\overline{\omega}_i, r_i - \sum_{h=1}^{\ell-1} p_h \overline{\omega}_{ih}\right)$ for $i = 1,\ldots,m-1$.

We point out that the Jacobian matrix of Π at $u = \left(p,(r_i),\left(\overline{\omega}_i\right)_{i=1}^{m-1},\left(\tilde{\xi}_i\right)\right)$ is a 2ml square matrix which has the following structure:

$$D\Pi(u) = \begin{bmatrix} \mathrm{A} & \mathrm{B} \\ 0 & Id \end{bmatrix}$$

Where A is the ml square submatrix of partial derivatives of the first ml components of Π with respect to the ml variables $\left(p_h\right)_{h=1}^{\ell-1},\left(r_i\right)_{i=1}^{m},\left(\overline{\omega}_{ih}\right)_{i=1,h=1}^{m-1,\ell-1}$, B the respective ml square submatrix of partial derivatives with respect to $\left(\tilde{\xi}_{i,h}\right)_{i=1,h=1}^{m,\ell}$, and finally Id and 0 are the ml identity and null matrix respectively. Thus A is also the Jacobian matrix of $\Pi^{\tilde{\xi}}$ at $\left(p,(r_i),\left(\overline{\omega}_i\right)_{i=1}^{m-1}\right)$.

Now, since (ω_i) is a regular economy in $\mathrm{E}^{\tilde{\xi}}$ and det $[\mathrm{A}]$ = det $[D\Pi(u)]$, we can deduce that $\left((\omega_i),\left(\tilde{\xi}_i\right)\right)$ is a regular economy in E and

$$\sum_{u\in\left(\Pi^{\tilde{\xi}}\right)^{-1}(\omega_i)} \text{sign }\left(\det\left[D\Pi^{\tilde{\xi}}(u)\right]\right)=\sum_{u\in\Pi^{-1}\left((\omega_i),\left(\tilde{\xi}_i\right)\right)} \text{sing }\left(\det\left[D\Pi(u)\right]\right),$$

which implies that

$$\deg\left(\Pi^{\tilde{\xi}}\right)=\deg(\Pi).$$

This ends the proof of Proposition 4.1.

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