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ON INCENTIVES IN THREE-SIDED MARKETS

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ON INCENTIVES IN THREE-SIDED MARKETS

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Abstract. In a class of three-sided matching problems that always have stable solutions, we show that no stable mechanism is strategy-proof for those who internalize the trilateral structure in their preferences. Furthermore, strong restrictions on preferences are needed to ensure that stability and one-sided strategy-proofness are compatible for all sides of the market. These results are related to the incompatibility between stability and one-sided group strategy-proofness in two-sided markets.

Keywords: Three-sided Matching Markets - Stability - Strategy-proofness
JEL Classification: D47, C78.

1. Introduction

In classical two-sided matching markets, a stable outcome always exists (Gale and Shapley, 1962) and for each side of the market there is a stable mechanism that is strategy-proof for its members (Dubins and Freedman, 1981; Roth, 1982). Although it is well-known that the inclusion of a third side may compromise the existence of stable matchings (Alkan, 1988; Ng and Hirschberg, 1991), the difficulties that may arise to ensure compatibility between stability and one-sided strategy-proofness have not been studied. This is the focus of the current paper.

We analyze the incentives to reveal information in three-sided problems with *mixed preferences* (Zhang and Zhong, 2021; Arenas and Torres-Martínez, 2023). Denoting by U, V, and W the sides of the market, we assume that agents in U have preferences defined on V, agents in V have preferences defined on W, and agents in W have lexicographic preferences defined on $V \times U$. Moreover, all sides of the market have the same number of agents. In this context, a *matching* is a distribution of the population in triplets formed by agents of different sides of the market such that every agent

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belongs to one and only one triplet. A matching is *stable* when no group of three agents may form a new triplet to improve the well-being of those who change their relevant partners.

In three-sided matching problems, a stable outcome may not exist even when agents ignore the trilateral structure of the market in their preferences (Lam and Plaxton, 2021; Lerner, 2022). As a consequence, there is a vast literature that studies restrictions on preference domains to ensure stability (cf., Danilov, 2003; Boros, Gurvich, Jaslar, and Krasner, 2004; Eriksson, Sjöstrand, and Strimling, 2006; Lahiri, 2009; Biró and McDermid, 2010; Huang, 2010; Manlove, 2013; Hofbauer, 2016; Zhang, Li, Fan, Shen, Shen, and Yu, 2019; Zhong and Bai, 2019; Bloch, Cantala, and Gibaja, 2020; Pashkovich and Poirrier, 2020; Raghavan, 2021). In some of these works, stable matchings are found through algorithms based on the sequential application of the *deferred acceptance mechanism* (Gale and Shapley, 1962). Following an analogous strategy, we characterize the stable matchings of a three-sided problem with mixed preferences (see Propositions 1 and 2).

Regarding incentives to reveal information, we show that no stable mechanism is strategy-proof for agents in W (see Theorem 1). Since there is a stable mechanism that is strategy-proof for agents in U and V (see Remark 1), it follows that only those who internalize the trilateral structure of the market in their preferences are capable of manipulating all stable mechanisms.

In our framework, an agent in W may have two reasons to misreport preferences when a stable mechanism is implemented: she may want to change her partner in V to a preferred one; or she may want to keep her partner in V but change the other pairs of $V \times W$ formed. This last strategy may improve her situation by inducing a redistribution of agents in U in order to maintain stability. Since agents in W have lexicographic preferences defined on $V \times U$, the first incentive to lie can be avoided when the triplets are formed in such a way that the pairs between agents in V and W are determined by the application of the *W-optimal stable mechanism*.[1] Indeed, in the associated marriage market between agents in V and W, this mechanism is strategy-proof for agents in W (see Dubins and Freedman, 1981; Roth, 1982). Hence, what our Theorem 1 shows is that the second incentive to lie is unavoidable. What happens is that strategy-proofness for agents in W is related to one-sided group strategy-proofness in two-sided markets (see Proposition 3). And in marriage markets, no stable mechanism is one-sided group strategy-proof (cf., Alcalde and Barberà, 1994).

We also show that a stable mechanism based on the sequential application of the Gale-Shapley deferred acceptance algorithm is strategy-proof for agents in W if and only if agents in V have acyclic preferences in the sense of Ergin (2002) (see Theorem 2). As it is well-known, Ergin acyclicity strongly restricts the heterogeneity of preferences. Therefore, substantial restrictions are needed to ensure that stability and one-sided strategy-proofness are compatible in our framework.

To illustrate our results, consider a centralized market for admissions to graduate programs and fellowship allocation. Assume that applicants prioritize graduate programs over fellowships, while graduate programs only rank applicants (no fellowships are awarded before admission). Funding sources have preferences for graduate programs because they want to promote certain areas of specialization. In this context, an applicant may wish to misreport her preferences when a stable

[1] This mechanism applies the *deferred acceptance algorithm* to the induced marriage market between agents in V and W assuming that agents in W make the proposals (see Gale and Shapley, 1962).

mechanism is implemented. For instance, a student may lower the rank of some attractive academic alternatives to free up places in even better programs. Additionally, to improve her fellowship, a student can prioritize some graduate programs in which she dominates other applicants, in order to change the distribution of places and thus reduce the demand for her preferred funding sources. Our main results ensure that these potential manipulations of preferences cannot be completely avoided unless graduate programs classify applicants fairly homogeneously.

The rest of the paper is organized as follows. Section 2 describes our model. Section 3 proves the existence of stable matchings. Section 4 studies the incentives to reveal information when a stable mechanism is implemented. Section 5 determines restrictions on preference domains to ensure that stability and one-sided strategy-proofness are compatible. Some proofs are collected in an Appendix.

2. Three-sided problems with mixed preferences

A *three-sided matching problem with mixed preferences*, represented by $[U, V, W, (\succ_h)_{h\in H}]$, is characterized by a set $H = U \cup V \cup W$ of agents and a preference profile $(\succ_h)_{h\in H}$ such that:

- The sets U, V, and W are disjoint and satisfy $|U| = |V| = |W|$.
- For each $u \in U$, $\succ_u$ is a linear order defined on V.[2]
- For each $v \in V$, $\succ_v$ is a linear order defined on W.
- For each $w \in W$, $\succ_w$ is a VU-lexicographic linear order defined on $V \times U$. That is, there is a linear order $\succ_{V,w}$ defined on V and a linear order $\succ_{U,w}$ defined on U such that

$$(v, u) \succ_w (v', u') \qquad \Longleftrightarrow \qquad [\, v \succ_{V,w} v' \,] \quad \text{or} \quad [\, v = v' \ \text{ and } \ u \succ_{U,w} u' \,].$$

  We refer to $(\succ_{V,w}, \succ_{U,w})$ as the linear orders representing $\succ_w$.

Let $\mathcal{R}$ be the collection of preference profiles $(\succ_h)_{h\in H}$ that satisfy the properties above.

A *matching* is a set $M \subseteq U \times V \times W$ such that any $h \in H$ belongs to one and only one triplet in M. Let $\mathcal{M}$ be the set of matchings. If a triplet (u, v, w) belongs to $M \in \mathcal{M}$, then the *relevant partners* of each member are denoted by $M(u) = v$, $M(v) = w$, and $M(w) = (v, u)$. A matching M is *blocked by a triplet* $(u, v, w) \in U \times V \times W$ when we have that:

$$v \succ_u M(u) \text{ or } v = M(u); \qquad w \succ_v M(v) \text{ or } w = M(v); \qquad (v, u) \succ_w M(w).$$

A matching is *stable* when it cannot be blocked by any triplet. Hence, in a stable matching no group of three agents of different sides of the market may deviate, forming a new triplet to improve the well-being of members who change their relevant partners.

We refer to any function $\Phi : \mathcal{R} \to \mathcal{M}$ as a *mechanism*. Moreover, it is said that:

- Φ is *stable* when $\Phi[(\succ_h)_{h\in H}]$ is stable in $[U, V, W, (\succ_h)_{h\in H}]$ for any $(\succ_h)_{h\in H} \in \mathcal{R}$.
- Φ is *strategy-proof for* $A \subseteq H$ when there is no agent $a \in A$ such that, for some preference profiles $(\succ_h)_{h\in H}, (\succ'_h)_{h\in H} \in \mathcal{R}$, $\Phi[(\succ_h)_{h\neq a}, \succ'_a](a) \succ_a \Phi[(\succ_h)_{h\in H}](a)$.

[2]A *linear order* is a complete, transitive, and strict preference relation.

Let $\mathcal{G} = [\mathcal{R}, \Phi]$ be the non-cooperative game in which agents report preferences $(\succ_h)_{h\in H}$ and the matching $\Phi[(\succ_h)_{h\in H}]$ is implemented. Notice that, the mechanism Φ is strategy-proof for agents in A if and only if it is a dominant strategy for them to report their true preferences in $\mathcal{G}$.

3. Existence of stable matchings

In this section, we show that any problem $[U, V, W, (\succ_h)_{h\in H}]$ has a stable matching.

Proposition 1. *Any three-sided matching problem with mixed preferences has a stable matching.*

Proof. Given $[U, V, W, (\succ_h)_{h\in H}]$, let $(\succ_{V,w}, \succ_{U,w})$ be the linear orders representing the preferences of $w \in W$. It follows from Gale and Shapley (1962, Theorem 1) that the two-sided matching market $[V, W, (\succ_v)_{v\in V}, (\succ_{V,w})_{w\in W}]$ has a stable matching. Hence, there is a bijective function $f : V \to W$ such that no $(v, w) \in V \times W$ satisfies $w \succ_v f(v)$ and $v \succ_{V,w} f^{-1}(w)$.

Let $Z = \{(v, w) \in V \times W : w = f(v)\}$. Given $z = (v, w) \in Z$, let $\succ_z^*$ be the linear order defined on U such that $u \succ_z^* u'$ if and only if $u \succ_{U,w} u'$. Moreover, given $u \in U$, let $\succ_u^*$ be the linear order defined on Z such that $z \succ_u^* z'$ if and only if $v \succ_u v'$, where $z = (v, w)$ and $z' = (v', w')$. Since Gale and Shapley (1962) ensures that $[U, Z, (\succ_h^*)_{h\in U\cup Z}]$ has a stable matching, there exists a bijective function $g : U \to Z$ such that there is no $(u, z) \in U \times Z$ satisfying $z \succ_u^* g(u)$ and $u \succ_z^* g^{-1}(z)$.

We claim that $M = \{(u, v, w) \in U \times V \times W : g(u) = (v, w)\}$ is stable in $[U, V, W, (\succ_h)_{h\in H}]$. By contradiction, suppose that (u^*, v^*, w^*) blocks M. Then, the following conditions hold:

- (a) $v^* \succ_{u^*} M(u^*)$ or $v^* = M(u^*)$.
- (b) $w^* \succ_{v^*} f(v^*)$ or $w^* = f(v^*)$.
- (c) $(v^*, u^*) \succ_{w^*} (f^{-1}(w^*), g^{-1}(f^{-1}(w^*), w^*))$.

Since $\succ_{w^*}$ is a VU-lexicographic linear order, the condition (c) is equivalent to requiring that either $v^* \succ_{V,w^*} f^{-1}(w^*)$ or $[v^* = f^{-1}(w^*)$ and $u^* \succ_{U,w^*} g^{-1}(f^{-1}(w^*), w^*)]$. Hence, as f determines a stable matching of $[V, W, (\succ_v)_{v\in V}, (\succ_{V,w})_{w\in W}]$, it follows from conditions (b)-(c) that

- (b') $w^* = f(v^*)$.
- (c') $u^* \succ_{U,w^*} g^{-1}(v^*, w^*)$.

If we denote $z^* = (v^*, w^*)$, then (c') is equivalent to $u^* \succ_{z^*}^* g^{-1}(z^*)$. Since g determines a stable matching of $[U, Z, (\succ_h^*)_{h\in U\cup Z}]$, and $v^* \succ_{u^*} M(u^*)$ is equivalent to $z^* \succ_{u^*}^* g(u^*)$, it follows from conditions (a) and (b') that $v^* = M(u^*)$ and $f(v^*) = w^*$. Hence $(u^*, v^*, w^*) \in M$, which is a contradiction. Therefore, the matching M is stable in $[U, V, W, (\succ_h)_{h\in H}]$.[3] □

For three-sided matching problems, there are many preference domain specifications under which a mechanisms based on the sequential application of the deferred acceptance algorithm is stable (Danilov, 2003; Manlove, 2013; Zhong and Bai, 2019; Bloch, Cantala, and Gibaja, 2020). The same property holds for three-sided problems with mixed preferences.

[3]To ensure the result of Proposition 1 it was crucial to restrict the preferences of agents in W to the domain of VU-lexicographic preferences (cf., Arenas and Torres-Martínez, 2023).

More precisely, let $\mathrm{DA}_{\mathrm{W},3} : \mathcal{R} \to \mathcal{M}$ be the mechanism that associates with each $(\succ_h)_{h\in H} \in \mathcal{R}$ the matching obtained through the following procedure:

Step 1. Given $w \in W$, let $(\succ_{V,w}, \succ_{U,w})$ be the linear orders representing $\succ_w$. Assuming that agents in W propose to agents in V, apply the *deferred acceptance algorithm* to the marriage market $[V, W, (\succ_v)_{v\in V}, (\succ_{V,w})_{w\in W}]$.
Let $Z \subseteq V \times W$ be the set of pairs formed.

Step 2. For each $z = (v, w) \in Z$, let $\succ^*_z$ be the linear order defined on U such that $u \succ^*_z u'$ whenever $u \succ_{U,w} u'$. Moreover, for each $u \in U$, let $\succ^*_u$ be the linear order defined on Z such that $z \succ^*_u z'$ as long as $v \succ_u v'$, where $z = (v, w)$ and $z' = (v', w')$.

Step 3. Assuming that agents in Z propose to agents in U, apply the *deferred acceptance algorithm* to the marriage market $[U, Z, (\succ^*_h)_{h\in U\cup Z}]$.
Define $\mathrm{DA}_{\mathrm{W},3}[(\succ_h)_{h\in H}]$ as the set of triplets obtained.

Notice that, the proof of Proposition 1 ensures that $\mathrm{DA}_{\mathrm{W},3}$ is a stable mechanism.

4. On stability and strategy-proofness

Unlike what happens in two-sided matching models (Dubins and Freedman, 1981; Roth, 1982), in three-sided matching markets with mixed preferences not all sides of the market have a stable mechanism that is strategy-proof for its members.

Theorem 1. *There is no stable mechanism $\Phi : \mathcal{R} \to \mathcal{M}$ that is strategy-proof for W.*

Proof. Consider a three-sided matching problem with mixed preferences $[U, V, W, (\succ_h)_{h\in H}]$ where the sets of agents are given by $U = \{u_1, u_2, u_3, u_4\}$, $V = \{v_1, v_2, v_3, v_4\}$, and $W = \{w_1, w_2, w_3, w_4\}$. Suppose that $(\succ_h)_{h\in U\cup V}$ satisfies the following conditions:[4]

$\succ_{u_1}$	$\succ_{u_2}$	$\succ_{u_3}$	$\succ_{u_4}$
v_1	v_2	v_2	v_4
⋮	v_3	v_3	⋮
⋮	⋮	⋮	⋮
⋮	⋮	⋮	⋮

$\succ_{v_1}$	$\succ_{v_2}$	$\succ_{v_3}$	$\succ_{v_4}$
w_2	w_1	w_3	w_2
w_3	w_2	⋮	w_4
w_1	⋮	⋮	⋮
w_4	⋮	⋮	⋮

Moreover, for each $w \in W$, the linear orders $(\succ_{V,w}, \succ_{U,w})$ representing $\succ_w$ are such that

$\succ_{V,w_1}$	$\succ_{V,w_2}$	$\succ_{V,w_3}$	$\succ_{V,w_4}$
v_1	v_2	v_3	v_4
v_2	v_1	⋮	⋮
⋮	⋮	⋮	⋮

$\succ_{U,w_1}$	$\succ_{U,w_2}$	$\succ_{U,w_3}$	$\succ_{U,w_4}$
u_1	u_1	u_2	u_4
u_3	u_2	u_3	⋮
⋮	⋮	⋮	⋮

The Proposition 2 (see Appendix) ensures that $[U, V, W, (\succ_h)_{h\in H}]$ has only two stable matchings:

$$\begin{aligned} M &= \{(u_3, v_2, w_1), (u_1, v_1, w_2), (u_2, v_3, w_3), (u_4, v_4, w_4)\}, \\ M' &= \{(u_1, v_1, w_1), (u_2, v_2, w_2), (u_3, v_3, w_3), (u_4, v_4, w_4)\}. \end{aligned}$$

[4] In the description of preferences, the vertical dots stand for arbitrary ordering of agents.

Therefore, if $\Phi : \mathcal{R} \to \mathcal{M}$ is a stable mechanism, we have two alternatives:

(i) $\Phi[(\succ_h)_{h\in H}] = M$. In this case, if all agents $h \neq w_2$ report their true preferences, then w_2 improves her situation by reporting $v_2 \succ^*_{V,w_2} v_4 \succ^*_{V,w_2} v_3 \succ^*_{V,w_2} v_1$ instead of $\succ_{V,w_2}$. Indeed, M' is the only stable matching in this scenario and $(v_2, u_2) \succ_{w_2} (v_1, u_1)$.

(ii) $\Phi[(\succ_h)_{h\in H}] = M'$. In this case, if all agents $h \neq w_3$ report their true preferences, then w_3 improves her situation by reporting $v_1 \succ^*_{V,w_3} v_3 \succ^*_{V,w_3} v_2 \succ^*_{V,w_3} v_4$ instead of $\succ_{V,w_3}$. Indeed, M is the only stable matching in this scenario and $(v_3, u_2) \succ_{w_3} (v_3, u_3)$.

We conclude that the stable mechanism Φ is not strategy-proof for W. □

To gain intuition about the arguments underlying the proof of Theorem 1, given a preference profile $(\succ_h)_{h\in H} \in \mathcal{R}$, assume that the *marriage market* $[V, W, (\succ_v)_{v\in V}, (\succ_{V,w})_{w\in W}]$ has only two stable matchings, denoted by μ and μ'. Without loss of generality, let μ be the V-optimal stable matching and μ' be the W-optimal stable matching.[5]

Gale and Sotomayor (1985, Theorem 1) ensures that at least one agent in W has incentives to misreport preferences when μ is implemented in the context of $[V, W, (\succ_v)_{v\in V}, (\succ_{V,w})_{w\in W}]$. Since agents in W have VU-lexicographic preferences, an analogous property holds in the context of the three-sided matching problem $[U, V, W, (\succ_h)_{h\in H}]$: given a stable mechanism $\Phi : \mathcal{R} \to \mathcal{M}$, at least one agent of W should have incentives to misrepresent preferences when $\Phi[(\succ_h)_{h\in H}](v) = \mu(v)$ for all $v \in V$. This is exactly what happens in the proof of Theorem 1 (see item (i)).

Therefore, to have any chance that Φ is strategy-proof for W, the projection of $\Phi[(\succ_h)_{h\in H}]$ on $V \times W$ needs to be equal to μ' (see Proposition 2 in the Appendix). However, if an agent $w \in W$ has the same partner in μ and μ', she may have incentives to misrepresent preferences in order to change the other couples of $V \times W$ without modifying her partner on V. Indeed, with this action she may improve her situation, by reducing the interest of some agents in U for the other pairs of $V \times W$. This is exactly what happens in the proof of Theorem 1 (see item (ii)).

In summary, it seems that strategy-proofness for W is related to one-sided strategy-proofness and one-sided non-bossiness in two-sided matching markets (see Proposition 3 in the Appendix).

Remark 1. There always exists a stable mechanism that is strategy-proof for those who only consider one side of the market in their preferences. Indeed, let $\mathrm{DA}_{\mathrm{VU},3} : \mathcal{R} \to \mathcal{M}$ be the mechanisms that associates with each $(\succ_h)_{h\in H}$ the matching that is obtained by the following procedure:

Step 1. Given $w \in W$, let $(\succ_{V,w}, \succ_{U,w})$ be the linear orders representing $\succ_w$. Assuming that agents in V propose to agents in W, apply the deferred acceptance algorithm to the marriage market $[V, W, (\succ_v)_{v\in V}, (\succ_{V,w})_{w\in W}]$.
Let $Z \subseteq V \times W$ be the set of pairs formed.

Step 2. For each $z = (v, w) \in Z$, let $\succ^*_z$ be the linear order defined on U such that $u \succ^*_z u'$ whenever $u \succ_{U,w} u'$. Also, for each $u \in U$, let $\succ^*_u$ be the linear order defined on Z such that $z \succ^*_u z'$ as long as $v \succ_u v'$, where $z = (v, w)$ and $z' = (v', w')$.

[5] That is, μ is weakly preferred by every agent in V to any other stable matching of $[V, W, (\succ_v)_{v\in V}, (\succ_{V,w})_{w\in W}]$, and the analogous property holds for μ' with respect to agents in W (see Gale and Shapley, 1962, Theorem 2).

Step 3. Assuming that agents in U make proposals, apply the *deferred acceptance algorithm* to the marriage market $[U, Z, (\succ_h^*)_{h\in U\cup Z}]$.
Define $\mathrm{DA}_{\mathrm{VU},3}[\succ_h)_{h\in H}]$ as the set of triplets obtained.

It follows from Gale and Shapley (1962, Theorem 1) and Proposition 2 that $\mathrm{DA}_{\mathrm{VU},3}[(\succ_h)_{h\in H}]$ is stable in $[U, V, W, (\succ_h)_{h\in H}]$. Since the deferred acceptance mechanism is strategy-proof for those that make proposals (Dubins and Freedman, 1981; Roth, 1982), the mechanism $\mathrm{DA}_{\mathrm{VU},3}$ is also strategy-proof for agents in $U \cup V$ in the preference domain $\mathcal{R}$. □

Some remarks are in order:

- One-sided Pareto efficiency dominates stability when it comes to finding a mechanism that is strategy-proof for W. Indeed, although no stable mechanism is strategy-proof for agents in W, using the serial dictatorship algorithm it is possible to find a mechanism that is both strategy-proof and Pareto efficient for agents in W (see Arenas and Torres-Martínez, 2022).
- In two-sided one-to-one matching problems, each side of the market has an optimal stable outcome (Gale and Shapley, 1962) and the set of stable matchings has a lattice structure (Knuth, 1976). These properties are lost in our context. Indeed, in the matching problem described in the proof of Theorem 1 there are only two stable outcomes, and none of them match each agent in W with the best partner that she may have in a stable matching.[6]

5. Existence of stable and strategy-proof mechanisms

In marriage markets, Ergin (2002) restricts preference domains to ensure the existence of stable and one-sided group strategy-proof mechanisms. In our framework, the same type of constraints will guarantee the existence of a stable mechanism that is strategy-proof for agents in W.

Given a domain of preferences $\mathcal{R}' \subseteq \mathcal{R}$, consider the following properties:

- $\mathcal{R}'$ is UW-*unrestricted* when for every $(\succ_h)_{h\in U\cup W}$ there exists $(\succ'_v)_{v\in V}$ such that
$$((\succ_u)_{u\in U}, (\succ'_v)_{v\in V}, (\succ_w)_{w\in W}) \in \mathcal{R}'.$$
- $\mathcal{R}'$ is V-*Ergin-acyclic* when there is no $(\succ_h)_{h\in H} \in \mathcal{R}'$ such that $w \succ_v w' \succ_v w''$ and $w'' \succ_{v'} w$ for some agents $v, v' \in V$ and $w, w', w'' \in W$.

Notice that, the preference profiles of the problems considered in the proof of Theorem 1 satisfy $w_2 \succ_{v_4} w_4 \succ_{v_4} w_3$ and $w_3 \succ_{v_3} w_2$. Hence, it seems that V-Ergin-acyclicity is necessary to ensure the existence of a stable mechanism that is strategy-proof for W.

Our last result determines the conditions that a preference domain must satisfy to guarantee that the stable mechanism $\mathrm{DA}_{\mathrm{W},3}$ is strategy-proof for W (see Section 3).

[6] While agents w_1 and w_2 prefer M' to M, agent w_3 prefers M to M'.

Theorem 2. *Let $\mathcal{R}' \subseteq \mathcal{R}$ be a* UW*-unrestricted preference domain. Then,* $\mathrm{DA}_{\mathrm{W},3} : \mathcal{R}' \to \mathcal{M}$ *is strategy-proof for W if and only if $\mathcal{R}'$ is* V*-Ergin-acyclic.*

The proof is given in the Appendix.

This result reinforces the relevance that the trilateral structure of the market has on agents' incentives to reveal information about preferences. Indeed, since V-Ergin-acyclicity substantially restricts the heterogeneity of the preferences of agents in V, we conclude that strong restrictions on $\mathcal{R}'$ are *necessary* to ensure that $\mathrm{DA}_{\mathrm{W},3} : \mathcal{R}' \to \mathcal{M}$ is strategy-proof for W.

6. Concluding remarks

We studied stability and incentives in a class of solvable three-sided matching problems. It was shown that no stable mechanism is strategy-proof for those that internalize the trilateral structure in their preferences (Theorem 1). Furthermore, this incompatibility between stability and strategy-proofness can be overcome only under strong restrictions on preferences (Theorem 2).

The study of incentives in multi-sided matching problems is still incipient (cf., Bloch, Cantala, and Gibaja, 2020, 2023). For this reason, it is natural to ask if our results hold in other classes of three-sided matching problems. Moreover, it is also interesting to analyze the incentives of agents to reveal information in multi-sided matching problems (cf., Sherstyuk, 1999; Ostrovsky, 2008; Hofbauer, 2016). The study of these topics is a matter for future research.

Appendix

On the structure of stable matchings. Let $\theta : \mathcal{M} \twoheadrightarrow U \times V$ and $\psi : \mathcal{M} \twoheadrightarrow V \times W$ be the projections of $\mathcal{M}$ on $U \times V$ and $V \times W$. That is, for each matching $M \in \mathcal{M}$ we have that

$$\theta(M) = \{(u,v) \in U \times V : M(u) = v\}, \qquad \psi(M) = \{(v,w) \in V \times W : M(v) = w\}.$$

The following result characterizes the structure of the set of stable matchings of a three-sided matching problem with mixed preferences.

Proposition 2. *A matching M is stable in $[U, V, W, (\succ_h)_{h \in H}]$ if and only if the following properties hold:*

(i) *The matching $\theta(M)$ is stable in the marriage market $[U, V, (\succ_u)_{u \in U}, (\succ_{U,M(v)})_{v \in V}]$.*
(ii) *The matching $\psi(M)$ is stable in the marriage market $[V, W, (\succ_v)_{v \in V}, (\succ_{V,w})_{w \in W}]$.*

Proof. Let $M \in \mathcal{M}$ be a stable matching of $[U, V, W, (\succ_h)_{h \in H}]$. The following arguments guarantee that properties (i) and (ii) hold:

- If $\theta(M)$ is unstable in $(U, V, (\succ_u)_{u \in U}, (\succ_{U,M(v)})_{v \in V})$, then there is a pair $(v,u) \in V \times U$ such that $v \succ_u \theta(M)(u)$ and $u \succ_{U,M(v)} \theta(M)(v)$.[7] It follows that $(u, v, M(v))$ blocks M. Indeed, when the triplet $(u, v, M(v))$ is formed, the situation of agent v does not change, $v \succ_u M(u)$, and $(v,u) \succ_{M(v)} M(M(v))$. This contradicts the stability of M.

[7] Given $M \in \mathcal{M}$ and $(u,v,w) \in M$, we denote $\theta(M)(u) = v$, $\theta(M)(v) = u$, $\psi(M)(v) = w$, and $\psi(M)(w) = v$.

- If $\psi(M)$ is unstable in $(V, W, (\succ_v)_{v\in V}, (\succ_{V,w})_{w\in W})$, then there is a pair $(v, w) \in V \times W$ such that $w \succ_v \psi(M)(v)$ and $v \succ_{V,w} \psi(M)(w)$. Let $u \in U$ such that $M(u) = v$. Then, the triplet (u, v, w) blocks M, because the situation of u does not change, $w \succ_v M(v)$, and $(v, u) \succ_w M(w)$. This contradicts the stability of the matching M.

On the other hand, it follows from the proof of Proposition 1 that $M \in \mathcal{M}$ is stable in the three-sided problem $[U, V, W, (\succ_h)_{h\in H}]$ as long as (i) and (ii) hold. $\square$

Three-sided markets vs. marriage markets. We will formalize the intuition that strategy-proofness for W in our framework is related to one-sided group strategy-proofness in marriage markets.

We need some notations for preference domains:

- Let $\mathcal{S}$ be the set of profiles $(S_h)_{h\in U\cup V}$ such that, for every $(u, v) \in U \times V$, S_u is a linear order defined on V and S_v is a linear order defined on U.
- Let $\mathcal{Q}$ be the set of profiles $(Q_h)_{h\in V\cup W}$ such that, for every $(v, w) \in V \times W$, Q_v is a linear order defined on W and Q_w is a linear order defined on V.
- Given $\mathcal{R}' \subseteq \mathcal{R}$, let $\mathcal{Q}(\mathcal{R}') \subseteq \mathcal{Q}$ be the set of preference profiles $(Q_h)_{h\in V\cup W}$ such that

$$((\succ_u)_{u\in U}, (Q_v)_{v\in V}, (Q_w, \succ_{U,w})_{w\in W}) \in \mathcal{R}'$$

for some linear orders $(\succ_u)_{u\in U}$ and $(\succ_{U,w})_{w\in W}$.

Notice that, for every preference profile $(\succ_h)_{h\in H} \in \mathcal{R}$ and for any matching N between the members in V and W, if $(\succ_{V,w}, \succ_{U,w})$ are the linear orders representing $\succ_w$, then

$$((\succ_u)_{u\in U}, (\succ_{U,N(v)})_{v\in V}) \in \mathcal{S} \qquad \wedge \qquad ((\succ_v)_{v\in V}, (\succ_{V,w})_{w\in W}) \in \mathcal{Q}.$$

Therefore, given $\Theta : \mathcal{S} \to \theta(\mathcal{M})$ and $\Psi : \mathcal{Q} \to \psi(\mathcal{M})$, we can define the mechanism $\Phi_{\Theta,\Psi} : \mathcal{R} \to \mathcal{M}$ that associates with each profile $(\succ_h)_{h\in H}$ the set of triplets (u, v, w) such that

$$\Theta[(\succ_u)_{u\in U}, (\succ_{U,N(v)})_{v\in V}](u) = v = \Psi[(\succ_v)_{v\in V}, (\succ_{V,w})_{w\in W}](w),$$

where $N = \Psi[(\succ_v)_{v\in V}, (\succ_{V,w})_{w\in W}]$. It follows from Proposition 2 that $\Phi_{\Theta,\Psi}$ is stable if and only if the mechanisms Θ and Ψ are stable. Moreover, Theorem 1 guarantees that the mechanism $\Phi_{\Theta,\Psi}$ cannot be stable and strategy-proof for W in the whole preference domain $\mathcal{R}$.

Given mechanisms $\Theta : \mathcal{S} \to \theta(\mathcal{M})$ and $\Psi : \mathcal{Q} \to \psi(\mathcal{M})$, it is said that:

- Θ is *strategy-proof for* V when there is no agent $v \in V$, preference profile $S \in \mathcal{S}$, and linear order S'_v defined on U such that $\Theta[S_{-v}, S'_v](v)\, S_v\, \Theta[S](v)$.
- Ψ is *strategy-proof for* W when there is no agent $w \in W$, preference profile $Q \in \mathcal{Q}$, and linear order Q'_w defined on V such that $\Psi[Q_{-w}, Q'_w](w)\, Q_w\, \Psi[Q](w)$.
- Ψ is *non-bossy for* W when there is no agent $w \in W$, preference profile $Q \in \mathcal{Q}$, and linear order Q'_w defined on V such that $\Psi[Q_{-w}, Q'_w](w) = \Psi[Q](w)$ and $\Psi[Q_{-w}, Q'_w] \neq \Psi[Q]$.
- Ψ is *group strategy-proof for* W when there is no group of agents $W' \subseteq W$, preference profile $Q \in \mathcal{Q}$, and linear orders $(Q'_w)_{w\in W'}$ defined on V such that:
  - For all $w \in W'$, either $\Psi[Q_{-w}, Q'_w](w)\, Q_w\, \Psi[Q](w)$ or $\Psi[Q_{-w}, Q'_w](w) = \Psi[Q](w)$.
  - For some $w \in W'$, we have that $\Psi[Q_{-w}, Q'_w](w)\, Q_w\, \Psi[Q](w)$.

Hence, a mechanism $\Psi : \mathcal{Q} \to \psi(\mathcal{M})$ is non-bossy for W as long as no agent $w \in W$ has incentives to misreport preferences in order to modify the situation of other agents without changing her own partner.

Notice that, Pápai (2020, Lemma 1) shows that a mechanism $\Psi : \mathcal{Q} \to \psi(\mathcal{M})$ is group strategy-proof for W if and only if it is both strategy-proof for W and non-bossy for W.

Proposition 3. *Let $\mathcal{R}' \subseteq \mathcal{R}$ be a* UW*-unrestricted sub-domain. Given stable mechanisms $\Theta : \mathcal{S} \to \theta(\mathcal{M})$ and $\Psi : \mathcal{Q}(\mathcal{R}') \to \psi(\mathcal{M})$, the following properties are equivalent:*

(i) *The mechanism $\Phi_{\Theta,\Psi} : \mathcal{R}' \to \mathcal{M}$ is strategy-proof for W.*

(ii) *Θ is strategy-proof for V and Ψ is group strategy-proof for W.*

Proof. The property that (i) implies (ii) is a consequence of the following three steps:

Step 1. *If $\Phi_{\Theta,\Psi} : \mathcal{R}' \to \mathcal{M}$ is strategy-proof for W, then $\Theta : \mathcal{S} \to \theta(\mathcal{M})$ is strategy-proof for V.*

Suppose that $\Theta : \mathcal{S} \to \theta(\mathcal{M})$ is not strategy-proof for V. Hence, there is an agent $\tilde{v} \in V$, a preference profile $S = (S_h)_{h\in U\cup V} \in \mathcal{S}$, and some linear order $S'_{\tilde{v}}$ defined on U such that $\Theta[S_{-\tilde{v}}, S'_{\tilde{v}}](\tilde{v})\ S_{\tilde{v}}\ \Theta[S](\tilde{v})$. Since $\mathcal{R}' \subseteq \mathcal{R}$ is a UW-unrestricted sub-domain, we can consider any preference profile $(\succ_h)_{h\in H} \in \mathcal{R}'$ which complies with following conditions:

- For each $u \in U$, $\succ_u = S_u$.
- For each $v \in V$, $\succ_v$ is an arbitrary linear order defined on W.
- For each $w \in W$, $\succ_w$ is represented by $(\succ_{V,w}, \succ_{U,w})$, where $\succ_{V,w}$ is an arbitrary linear order defined on V, $\succ_{U,w} = S_{N(w)}$, and $N = \Psi[(\succ_v)_{v\in V}, (\succ_{V,w})_{w\in W}]$.

Let $\tilde{w} = N(\tilde{v})$ and $\succ'_{\tilde{w}}$ be a VU-lexicographic linear order represented by $(\succ_{V,\tilde{w}}, S'_{\tilde{v}})$. Then, the property $\Theta[S_{-\tilde{v}}, S'_{\tilde{v}}](\tilde{v})\ S_{\tilde{v}}\ \Theta[S](\tilde{v})$ can be rewritten as $\theta(\Phi_{\Theta,\Psi}[(\succ_h)_{h\neq\tilde{w}}, \succ'_{\tilde{w}}])(\tilde{v}) \succ_{U,\tilde{w}} \theta(\Phi_{\Theta,\Psi}[(\succ_h)_{h\in H}])(\tilde{v})$.

The definition of $\Phi_{\Theta,\Psi}$ ensures that $\psi(\Phi_{\Theta,\Psi}[(\succ_h)_{h\neq\tilde{w}}, \succ'_{\tilde{w}}])(\tilde{w}) = \psi(\Phi_{\Theta,\Psi}[(\succ_h)_{h\in H}])(\tilde{w})$. Since $\tilde{v} = N(\tilde{w})$ and $\succ_{\tilde{w}}$ is VU-lexicographic, it follows that $\Phi_{\Theta,\Psi}[(\succ_h)_{h\neq\tilde{w}}, \succ'_{\tilde{w}}](\tilde{w}) \succ_{\tilde{w}} \Phi_{\Theta,\Psi}[(\succ_h)_{h\in H}](\tilde{w})$.

Thus, the mechanism $\Phi_{\Theta,\Psi}$ is not strategy-proof for W. *Q.E.D.*

Step 2. *If $\Phi_{\Theta,\Psi} : \mathcal{R}' \to \mathcal{M}$ is strategy-proof for W, then $\Psi : \mathcal{Q}(\mathcal{R}') \to \psi(\mathcal{M})$ is strategy-proof for W.*

Suppose that $\Psi : \mathcal{Q}(\mathcal{R}') \to \psi(\mathcal{M})$ is not strategy-proof for W. Hence, there is an agent $\tilde{w} \in W$, a preference profile $Q = (Q_h)_{h\in V\cup W} \in \mathcal{Q}(\mathcal{R}')$, and some linear order $Q'_{\tilde{w}}$ defined on V such that

$$(Q_{-\tilde{w}}, Q'_{\tilde{w}}) \in \mathcal{Q}(\mathcal{R}') \qquad \text{and} \qquad \Psi[Q_{-\tilde{w}}, Q'_{\tilde{w}}](\tilde{w})\ Q_{\tilde{w}}\ \Psi[Q](\tilde{w}).$$

By the definition of $\mathcal{Q}(\mathcal{R}')$, we can consider any $(\succ_h)_{h\in H} \in \mathcal{R}'$ satisfying the following conditions:

- For each $u \in U$, $\succ_u$ is an arbitrary linear order defined on V.
- For each $v \in V$, $\succ_v = Q_v$.
- For each $w \in W$, $\succ_w$ is represented by $(\succ_{V,w}, \succ_{U,w})$, where $\succ_{V,w} = Q_w$ and $\succ_{U,w}$ is an arbitrary linear order defined on U.

Let $\succ'_{\tilde{w}}$ be a VU-lexicographic linear order represented by $(Q'_{\tilde{w}}, \succ'_{U,\tilde{w}})$, where $\succ'_{U,\tilde{w}}$ is an arbitrary linear order defined on U. Then, the property $\Psi[Q_{-\tilde{w}}, Q'_{\tilde{w}}](\tilde{w})\ Q_{\tilde{w}}\ \Psi[Q](\tilde{w})$ can be rewritten as

$$\psi(\Phi_{\Theta,\Psi}[(\succ_h)_{h\neq\tilde{w}}, \succ'_{\tilde{w}}])(\tilde{w}) \succ_{V,\tilde{w}} \psi(\Phi_{\Theta,\Psi}[(\succ_h)_{h\in H}])(\tilde{w}).$$

Since $\succ_{\tilde{w}}$ is VU-lexicographic, the definition of ψ implies that $\Phi_{\Theta,\Psi}[(\succ_h)_{h\neq\tilde{w}}, \succ'_{\tilde{w}}](\tilde{w}) \succ_{\tilde{w}} \Phi_{\Theta,\Psi}[(\succ_h)_{h\in H}](\tilde{w})$. Thus, the mechanism $\Phi_{\Theta,\Psi}$ is not strategy-proof for W. *Q.E.D.*

Step 3. *If $\Phi_{\Theta,\Psi} : \mathcal{R}' \to \mathcal{M}$ is strategy-proof for W, then $\Psi : \mathcal{Q}(\mathcal{R}') \to \psi(\mathcal{M})$ is non-bossy for W.*

Suppose that $\Psi : \mathcal{Q}(\mathcal{R}') \to \psi(\mathcal{M})$ is bossy for W. Hence, there is an agent $\tilde{w} \in W$, a preference profile $Q = (Q_h)_{h\in V\cup W} \in \mathcal{Q}(\mathcal{R}')$, and some linear order $Q'_{\tilde{w}}$ defined on V such that $(Q_{-\tilde{w}}, Q'_{\tilde{w}}) \in \mathcal{Q}(\mathcal{R}')$,

$$\tilde{v} \equiv \Psi[Q_{-\tilde{w}}, Q'_{\tilde{w}}](\tilde{w}) \;=\; \Psi[Q](\tilde{w}) \qquad \wedge \qquad \Psi[Q_{-\tilde{w}}, Q'_{\tilde{w}}] \;\neq\; \Psi[Q].$$

Since $|V| = |W|$, $\Psi[Q_{-\tilde{w}}, Q'_{\tilde{w}}] \neq \Psi[Q]$ implies that there are agents $v_1 \in V$ and $w_1, w_2 \in W \setminus \{\tilde{w}\}$ such that $w_1 \equiv \Psi[Q](v_1) \neq \Psi[Q_{-\tilde{w}}, Q'_{\tilde{w}}](v_1) \equiv w_2$. Moreover, as $\mathcal{R}' \subseteq \mathcal{R}$ is a UW-unrestricted sub-domain, if we denote $U = \{u_1, \ldots, u_{n-1}, \tilde{u}\}$, $V = \{v_1, \ldots, v_{n-1}, \tilde{v}\}$ and $W = \{w_1, \ldots, w_{n-1}, \tilde{w}\}$, the definition of $\mathcal{Q}(\mathcal{R}')$ implies that any preference profile $(\succ_h)_{h\in H}$ satisfying the following conditions belongs to $\mathcal{R}'$:

- The linear orders $\succ_{u_1}$, $\succ_{u_2}$, and $\succ_{\tilde{u}}$ are such that

$\succ_{u_1}$	$\succ_{u_2}$	$\succ_{\tilde{u}}$
v_1	v_1	$\tilde{v}$
$\tilde{v}$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$

- For $u \in U \setminus \{u_1, u_2, \tilde{u}\}$, $\succ_u$ is an arbitrary linear order defined on V.
- For each $v \in V$, $\succ_v = Q_v$.
- For each $w \in W$ there is a linear order $\succ_{U,w}$ defined on U such that $\succ_w$ is represented by $(Q_w, \succ_{U,w})$, where $\succ_{U,w_1}$, $\succ_{U,w_2}$, and $\succ_{U,\tilde{w}}$ are such that

$\succ_{U,w_1}$	$\succ_{U,w_2}$	$\succ_{U,\tilde{w}}$
u_1	u_2	u_1
$\vdots$	$\vdots$	$\tilde{u}$
$\vdots$	$\vdots$	$\vdots$

Let $\succ'_{\tilde{w}}$ be the VU-lexicographic linear order represented by $(Q'_{\tilde{w}}, \succ_{U,\tilde{w}})$. It follows that the properties $\Psi[Q_{-\tilde{w}}, Q'_{\tilde{w}}](\tilde{w}) = \Psi[Q](\tilde{w})$ and $\Psi[Q_{-\tilde{w}}, Q'_{\tilde{w}}] \neq \Psi[Q]$ can be rewritten as

$$\begin{aligned} \psi(\Phi_{\Theta,\Psi}[(\succ_h)_{h\neq\tilde{w}}, \succ'_{\tilde{w}}])(\tilde{w}) \;&=\; \psi(\Phi_{\Theta,\Psi}[(\succ_h)_{h\in H}])(\tilde{w}), \\ \psi(\Phi_{\Theta,\Psi}[(\succ_h)_{h\neq\tilde{w}}, \succ'_{\tilde{w}}]) \;&\neq\; \psi(\Phi_{\Theta,\Psi}[(\succ_h)_{h\in H}]). \end{aligned}$$

Let $M = \Phi_{\Theta,\Psi}[(\succ_h)_{h\in H}]$ and $M' = \Phi_{\Theta,\Psi}[(\succ_h)_{h\neq\tilde{w}}, \succ'_{\tilde{w}}]$.

Since $M(v_1) = w_1$ and $M'(v_1) = w_2$, the definitions of $(\succ_h)_{h\in H}$ and $\succ'_{\tilde{w}}$ ensure that $\tilde{v}$ and $\tilde{u}$ form a couple in any stable matching of the marriage market $(U, V, (\succ_u)_{u\in U}, (\succ_{U,M(v)})_{v\in V})$. Analogously, $\tilde{v}$ and u_1 form a couple in any stable matching of $(U, V, (\succ_u)_{u\in U}, (\succ_{U,M'(v)})_{v\in V})$. Hence, the Proposition 2 implies that $\Phi_{\Theta,\Psi}[(\succ_h)_{h\in H}](\tilde{w}) = (\tilde{v}, \tilde{u})$ and $\Phi_{\Theta,\Psi}[(\succ_h)_{h\neq\tilde{w}}, \succ'_{\tilde{w}}](\tilde{w}) = (\tilde{v}, u_1)$. Therefore, the mechanism $\Phi_{\Theta,\Psi}$ is not strategy-proof for W because $\Phi_{\Theta,\Psi}[(\succ_h)_{h\neq\tilde{w}}, \succ'_{\tilde{w}}](\tilde{w}) \succ_{\tilde{w}} \Phi_{\Theta,\Psi}[(\succ_h)_{h\in H}](\tilde{w})$. *Q.E.D.*

To show that (ii) implies (i), assume that $\Theta : \mathcal{S} \to \theta(\mathcal{M})$ is strategy-proof for V and $\Psi : \mathcal{Q}(\mathcal{R}') \to \psi(\mathcal{M})$ is group strategy-proof for W (equivalently, Ψ is strategy-proof for W and non-bossy for W).

By contradiction, assume that $\Phi_{\Theta,\Psi} : \mathcal{R}' \to \mathcal{M}$ is not strategy-proof for W. Hence, there is an agent $\tilde{w} \in W$, a preference profile $(\succ_h)_{h\in H} \in \mathcal{R}'$ and a linear order $\succ'_{\tilde{w}}$ such that $((\succ_h)_{h\neq\tilde{w}}, \succ'_{\tilde{w}}) \in \mathcal{R}'$ and

$$(v', u') \equiv \Phi_{\Theta,\Psi}[(\succ_h)_{h\neq\tilde{w}}, \succ'_{\tilde{w}}](\tilde{w}) \succ_{\tilde{w}} \Phi_{\Theta,\Psi}[(\succ_h)_{h\in H}](\tilde{w}) \equiv (\tilde{v}, \tilde{u}).$$

If $(\succ_{V,w}, \succ_{U,w})$ are the linear orders representing $\succ_w$, consider the profile $Q = (Q_h)_{h \in V \cup W} \in \mathcal{Q}(\mathcal{R}')$ such that $Q_v = \succ_v$ for all $v \in V$, $Q_w = \succ_{V,w}$ for all $w \in W$. Also, if $(\succ'_{V,\tilde{w}}, \succ'_{U,\tilde{w}})$ are the linear orders representing $\succ'_{\tilde{w}}$, let $Q'_{\tilde{w}}$ be a linear order defined on V such that $Q'_{\tilde{w}} = \succ'_{V,\tilde{w}}$.

There are two relevant cases to analyze:

- *Assume that $v' \neq \tilde{v}$.* Since $\succ_{\tilde{w}}$ is VU-lexicographic, it follows that $v' \succ_{V,\tilde{w}} \tilde{v}$. Moreover, by the definition of $\Phi_{\Theta,\Psi}$, we have that $\succ'_{V,\tilde{w}} \neq \succ_{V,\tilde{w}}$. Since the property $v' \succ_{V,\tilde{w}} \tilde{v}$ is equivalent to

  $$v' = \psi(\Phi_{\Theta,\Psi}[(\succ_h)_{h \neq \tilde{w}}, \succ'_{\tilde{w}}])(\tilde{w}) \ \succ_{V,\tilde{w}} \ \psi(\Phi_{\Theta,\Psi}[(\succ_h)_{h \in H}])(\tilde{w}) = \tilde{v},$$

  by the definition of ψ we have that $\Psi[Q_{-\tilde{w}}, Q'_{\tilde{w}}](\tilde{w}) \, Q_{\tilde{w}} \, \Psi[Q](\tilde{w})$. Therefore, the mechanism Ψ is not strategy-proof for W. A contradiction.

- *Assume that $v' = \tilde{v}$ and $u' \neq \tilde{u}$.* Since $\succ_{\tilde{w}}$ is VU-lexicographic, it follows that $u' \succ_{U,\tilde{w}} \tilde{u}$. There exist two possibilities depending of the linear orders $(\succ'_{V,\tilde{w}}, \succ'_{U,\tilde{w}})$ representing $\succ'_{\tilde{w}}$:
  - *Suppose that $\succ'_{V,\tilde{w}} \neq \succ_{V,\tilde{w}}$ and $\succ'_{U,\tilde{w}} = \succ_{U,\tilde{w}}$.* Since $u' \neq \tilde{u}$ and $\Psi[Q_{-\tilde{w}}, Q'_{\tilde{w}}](\tilde{w}) = \Psi[Q](\tilde{w})$, it follows from the definition of $\Phi_{\Theta,\Psi}$ that there is $\hat{w} \in W$ such that $\Psi[Q_{-\tilde{w}}, Q'_{\tilde{w}}](\hat{w}) \neq \Psi[Q](\hat{w})$. Thus, the mechanism Ψ is bossy for W. A contradiction.
  - *Suppose that $\succ'_{U,\tilde{w}} \neq \succ_{U,\tilde{w}}$.* If $M = \Phi_{\Theta,\Psi}[(\succ_h)_{h \in H}]$, let $S = (S_h)_{h \in U \cup V} \in \mathcal{S}$ be such that $S_u = \succ_u$ for all $u \in U$, $S_v = \succ_{U,M(v)}$ for all $v \in V$, and $S'_{\tilde{v}} = \succ'_{U,M(\tilde{v})}$. Since $u' \succ_{U,\tilde{w}} \tilde{u}$ can be rewritten as $u' = \theta(\Phi_{\Theta,\Psi}[(\succ_h)_{h \neq \tilde{w}}, \succ'_{\tilde{w}}])(\tilde{v}) \ \succ_{U,\tilde{w}} \ \theta(\Phi_{\Theta,\Psi}[(\succ_h)_{h \in H}])(\tilde{v}) = \tilde{u}$, it follows from the definition of θ that $\Theta[S_{-\tilde{v}}, S'_{\tilde{v}}](\tilde{v}) \, S_{\tilde{v}} \, \Theta[S](\tilde{v})$. This contradicts the strategy-proofness of Θ.

It follows that $(v', u') = (\tilde{v}, \tilde{u})$, which is incompatible with our assumption that $(v', u') \succ_{\tilde{w}} (\tilde{v}, \tilde{u})$.

Therefore, the mechanism $\Phi_{\Theta,\Psi} : \mathcal{R}' \to \mathcal{M}$ is strategy-proof for W. $\square$

Proof of Theorem 2. Let $\Theta : \mathcal{S} \to \theta(\mathcal{M})$ be the mechanism that associates with each $(S_h)_{h \in U \cup V}$ the V-optimal stable matching of the marriage market $[U, V, (S_h)_{h \in U \cup V}]$. Also, let $\Psi : \mathcal{Q}(\mathcal{R}') \to \psi(\mathcal{M})$ be the mechanism that associates with each $(Q_h)_{h \in V \cup W} \in \mathcal{Q}(\mathcal{R}')$ the W-optimal stable matching of the marriage market $[V, W, (Q_h)_{h \in V \cup W}]$. Notice that $\mathrm{DA}_{\mathrm{W},3} \equiv \Phi_{\Theta,\Psi}$ in the sub-domain $\mathcal{R}'$.

$[(i) \Longrightarrow (ii)]$ Suppose that the mechanism $\mathrm{DA}_{\mathrm{W},3} : \mathcal{R}' \to \mathcal{M}$ is strategy-proof for W. Since $\mathcal{R}' \subseteq \mathcal{R}$ is UW-unrestricted, Proposition 3 ensures that $\Psi : \mathcal{Q}(\mathcal{R}') \to \psi(\mathcal{M})$ is both strategy-proof for W and non-bossy for W. Hence, it follows from Ergin (2002, Theorem 1) that, for any preference profile $(Q_h)_{h \in V \cup W} \in \mathcal{Q}(\mathcal{R}')$, the linear orders $(Q_v)_{v \in V}$ are Ergin-acyclic in the sense that there are no $v, v' \in V$ and $w, w', w'' \in W$ such that $w Q_v w' Q_v w''$ and $w'' Q_{v'} w$.[8] Therefore, the sub-domain $\mathcal{R}'$ is V-Ergin-acyclic.

$[(ii) \Longrightarrow (i)]$ When $\mathcal{R}'$ is UW-unrestricted and V-Ergin-acyclic, Ergin (2002, Theorem 1) ensures that $\Psi : \mathcal{Q}(\mathcal{R}') \to \psi(\mathcal{M})$ is both strategy-proof for W and non-bossy for W (cf., Narita, 2021). Moreover, the mechanism $\Theta : \mathcal{S} \to \theta(\mathcal{M})$ is strategy-proof for V (see Dubins and Freedman, 1981, Theorem 9; Roth, 1982, Theorem 5). Hence, Proposition 3 guarantees that $\mathrm{DA}_{\mathrm{W},3}$ is strategy-proof for W in $\mathcal{R}'$. $\square$

[8]Although in our framework agents in W consider all agents in V admissible, the main result of Ergin (2002) can be easily adapted. Admissibility plays a role only in steps $[(ii) \Longrightarrow (iv)]$ and $[(iii) \Longrightarrow (iv)]$ of the proof of his Theorem 1. In these steps, given $v_1, v_2 \in V$ and $w_1, w_2, w_3 \in W$, it is considered a preference profile $(Q_w)_{w \in W}$ such that $v_2 Q_{w_1} v_1 Q_{w_1} w_1 Q_{w_1} \cdots$, $v_1 Q_{w_2} w_2 Q_{w_2} \cdots$, $v_1 Q_{w_3} v_2 Q_{w_3} w_3 Q_{w_3} \cdots$, and $w Q_w v$ for all $w \in W \setminus \{w_1, w_2, w_3\}$ and $v \in V$. However, if $\{v_3, \ldots, v_n\} \equiv V \setminus \{v_1, v_2\}$ and $\{w_4, \ldots, w_n\} \equiv W \setminus \{w_1, w_2, w_3\}$, the same implications that are obtained from $(Q_w)_{w \in W}$ can be ensured by working with the linear orders $(\tilde{Q}_w)_{w \in W}$ defined on V and characterized by $v_2 \tilde{Q}_{w_1} v_1 \tilde{Q}_{w_1} \cdots$, $v_1 \tilde{Q}_{w_2} v_3 \tilde{Q}_{w_2} \cdots$, $v_1 \tilde{Q}_{w_3} v_2 \tilde{Q}_{w_3} \cdots$, and $v_j \tilde{Q}_{w_j} \cdots$ for all $j \in \{4, \ldots, n\}$.

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