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ON EQUILIBRIUM EXISTENCE WITH ENDOGENOUS RESTRICTED FINANCIAL PARTICIPATION

Abdelkrim Seghir
Juan Pablo Torres-Martínez

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ABDELKRIM SEGHIR AND JUAN PABLO TORRES-MARTÍNEZ

Abstract. Allowing for durable commodities, we prove equilibrium existence in an abstract incomplete market economy with endogenous restricted financial participation without requiring financial survival assumptions. We apply our results to general financial structures including nominal, real and collateralized asset markets.

Keywords. Equilibrium, Incomplete markets, Endogenous restricted financial participation.

JEL classification. D52, D54.

1. Introduction

Modern financial markets restrict agents' participation in terms of which assets they can trade. Collateral requirements, student loans, privileges for first home buyers, different countries with different access to credits for political reasons are few examples of possible financial restrictions. The objective of this paper is to study restricted financial participations with incomplete markets and durable goods for a general financial structure.

In the literature, there are two ways of modeling financial participation restrictions. The first one assumes that the restrictions are exogenously given. For such a framework, Angeloni and Cornet (2006) prove equilibrium existence for real financial markets assuming that portfolio sets are convex and compact, containing a neighborhood of zero at least for some agents (this last requirement is also called *financial survival assumption*). More recently, Aouani and Cornet (2009) show equilibrium existence with restricted financial participation for the numeraire and the nominal cases under a *nonredundancy-type hypothesis* assuming that portfolio sets are closed vector spaces containing a neighborhood of zero.[1] Moreover, Cornet and Gopalan (2010) show equilibrium existence for nominal financial markets, also assuming that agents' portfolio sets are closed and convex sets containing zero, but instead of survival assumption they impose a *spanning condition* on the set of admissible

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[1] In the case of nominal assets and unrestricted participation, the nonredundancy-type assumption is equivalent to the classical hypothesis that the payoff matrix has full rank.

portfolios: the closed cone generated by the union of portfolio sets is a linear space. The second way of modeling financial participation restrictions is to assume that these constraints emerge endogenously in the model due to regulatory, institutional or budgetary considerations, that may depend on markets prices or commodity purchases. Such a structure was considered by Cass, Siconolfi and Villanacci (2001) and more recently by Carosi, Gori and Villanacci (2009). Cass, Siconolfi and Villanacci (2001) prove equilibrium existence for nominal assets, where portfolio sets are described by restriction functions that depend only on asset prices and satisfy some differentiability and regularity assumptions. Carosi, Gori and Villanacci (2009) show equilibrium existence for numeraire financial markets, where restricted participations are given by functions that depend on commodity and asset prices and satisfying some homogeneity, differentiability and regularity assumptions.

In our model, restricted financial participations are endogenous, in the sense that they may depend on commodity purchases, as in the case of mortgage markets where physical guarantees need to be held to obtain a loan. More precisely, portfolio participation constraints are represented by a general correspondence whose values are not necessarily given by inequalities determined by differentiable or regular functions. Without imposing either survival financial assumptions or linear spanning conditions over financial spaces, we prove equilibrium existence in an *abstract economy* where admissible debts belong to a compact set and utility functions are unbounded. The former assumption will be endogenously satisfied in our applications, for instance, to show the existence of equilibrium with endogenously restricted financial participation in either nominal asset markets with non-redundant assets or real assets markets where short-sales are endogenously bounded. Since we allow portfolio constraints to depend on purchases of commodities, we can apply our main result to extend the seminal model of collateralized asset markets of Dubey, Geanakoplos and Zame (1995) and Geanakoplos and Zame (1997, 2002, 2007) to allow for endogenous restricted participation. As we do not impose any financial survival assumption, the presence of exclusive collateralized loans, i.e., credit opportunities that may be negotiated only by some agents, is compatible with equilibrium.

The remaining of the paper is organized as follows. Section 2 is devoted to present our abstract economy and to state the associated equilibrium existence theorem. In Section 3 we apply this result to extend the classical models of nominal, real and collateralized assets to allow for restricted financial participation. Technical proof are given in an appendix.

2. An abstract financial economy

We consider an exchange economy with two periods $t = 0$ and $t = 1$ and uncertainty about which state of nature s of a finite set $\mathcal{S} := \{1, \dots, S\}$ will prevail at $t = 1$. Let $s = 0$ denote the state of nature (known with certainty) at period $t = 0$ and let $\mathcal{S}^* = \{0\} \cup \mathcal{S}$ be the set of all states of nature in the economy.

There is a set $\mathcal{L} = \{1, \ldots, L\}$ of perfectly divisible commodities that can be traded in spot markets at any state of nature $s \in \mathcal{S}^*$. The commodity space is $\mathbb{R}_+^{L(S+1)}$ and $p = (p_s; s \in S^*)$ denotes the plan of unitary commodity prices. We allow for depreciation, durability and transformation of commodities into other goods between periods. Thus, we assume that any bundle x consumed at the first period is transformed into a bundle $Y_s\, x$ at state of nature $s \in \mathcal{S}$, where Y_s is an $(L \times L)-$ matrix with non-negative entries.

In addition to commodity markets, there is a financial market that consists of a finite set $\mathcal{J} = \{1, \ldots, J\}$ of assets. Each asset $j \in \mathcal{J}$ can be traded at the first period and delivers a random return across the states of nature at the second period. More precisely, each unit of contract $j \in \mathcal{J}$ promises to deliver, at each state $s \in \mathcal{S}$, a financial return $V_j^s(p_s) \in \mathbb{R}_+$. That is, each asset j is characterized by a vector map $p \longrightarrow (V_j^s(p_s); s \in \mathcal{S})$ and a unitary price $q_j \in \mathbb{R}_+$. Let us denote by $q = (q_j; j \in \mathcal{J})$ the vector of unitary asset prices, and by $V : \mathbb{R}_+^{L(S+1)} \to \mathbb{R}_+^{S\times J}$ the map that associates to each p the vector $V(p) = (V_j^s(p_s); (s, j) \in \mathcal{S} \times \mathcal{J})$.

There is a finite number H of agents. Each agent $h \in \mathcal{H} = \{1, \ldots, H\}$ is characterized by a consumption space $X^h = \mathbb{R}_+^{L(S+1)}$, a utility function $u^h : X^h \to \mathbb{R}$, and physical endowments $w^h = (w_s^h; s \in S^*) \in \mathbb{R}_+^{L(S+1)}$. Agent h's vector of accumulated endowments is denoted by $W^h := (W_0^h, (W_s^h; s \in \mathcal{S})) = (w_0^h, (w_s^h + Y_s\, w_0^h; s \in \mathcal{S})) \in \mathbb{R}_+^{L(S+1)}$.

At $t = 0$, each agent $h \in \mathcal{H}$ chooses a portfolio $\theta^h - \varphi^h$, where $\theta^h = (\theta_j^h;\ j \in \mathcal{J}) \in \mathbb{R}_+^J$ are the quantities of assets that agent h buys and $\varphi^h = (\varphi_j^h;\ j \in \mathcal{J}) \in \mathbb{R}_+^J$ are the quantities of assets that he sells. In addition, at each state of nature $s \in \mathcal{S}^*$, agent h chooses a consumption bundle x_s^h.

In our model, agents' financial positions may be restricted, in the sense that, each agent h is constrained to choose short-sales $\varphi^h \in \Phi^h(x_0^h) \subset \mathbb{R}_+^J$, where the correspondence $\Phi^h : \mathbb{R}_+^L \twoheadrightarrow \mathbb{R}_+^J$ associates first period commodity purchases with admissible debts. Thus, we allow credit opportunities to depend on commodity purchases.

Note that, since survival assumptions and spanning conditions over admissible portfolio sets are not required, agents may have access only to some credit contracts. That is, there may exist a set of canonical vectors of $\mathbb{R}^J$, $A = \{e(j); j \in \mathcal{J}'\}$, where $\mathcal{J}' \subset \mathcal{J}$, such that $\Phi^h(x_0^h) \cap < A >= \emptyset$, for some $x_0^h \in \mathbb{R}_+^L$.[2]

Given prices (p, q), the budget set $B^h(p, q)$ of agent $h \in \mathcal{H}$ is the set of plans $(x^h, \theta^h, \varphi^h) \in \mathbb{E} := X^h \times \mathbb{R}_+^J \times \mathbb{R}_+^J$ such that $\varphi^h \in \Phi^h(x_0^h)$ and

$$p_0 x_0^h + \sum_{j \in \mathcal{J}} q_j(\theta_j^h - \varphi_j^h) \leq p_0 w_0^h; \qquad\qquad p_s x_s^h \leq p_s w_s^h + p_s Y_s x_0^h + \sum_{j \in J} V_j^s(p_s)(\theta_j^h - \varphi_j^h).$$

[2]The set $< A >$ denotes the linear space generated by A.

DEFINITION. *An equilibrium of our economy is a vector of prices* $(\overline{p},\overline{q}) \in \mathbb{R}^L_+ \times \mathbb{R}^J_+$ *and allocations* $((\overline{x}^h,\overline{\theta}^h,\overline{\varphi}^h); h \in \mathcal{H}) \in \mathbb{E}^H$ *such that:*

(i) *For each agent* $h \in \mathcal{H}$, $(\overline{x}^h,\overline{\theta}^h,\overline{\varphi}^h) \in$ *Argmax* $\left\{u^h(x^h);(x^h,\theta^h,\varphi^h) \in B^h(\overline{p},\overline{q})\right\}$.

(ii) *Physical and asset markets clearing conditions hold,*

$$\sum_{h\in\mathcal{H}} (\overline{x}^h,\overline{\varphi}^h) = \sum_{h\in H} (W^h,\overline{\theta}^h).$$

Our equilibrium existence result is:

THEOREM. *Suppose that the following assumptions hold:*

(A1) *For each* $h \in \mathcal{H}$, $u^h : X^h \to \mathbb{R}$ *is continuous, strongly quasi-concave and strictly increasing.*[3]

(A2) *For each agent* $h \in \mathcal{H}$ *there is* $l(h) \in \mathcal{L}$ *such that,*

$$\lim_{x^h_{0,l(h)}\to+\infty} u^h(x^h_0,(x^h_s; s \in S)) = +\infty, \quad \forall (x^h_s; s \in \mathcal{S}^*) \in \mathbb{R}^{L\times\mathcal{S}^*}_{++}.$$

(A3) *For each* $h \in \mathcal{H}$, *accumulated endowments* $W^h \in \mathbb{R}^L_{++}$.

(A4) *The map* $V(p) = (V^s_j(p_s);(s,j) \in \mathcal{S}\times\mathcal{J})$ *is continuous. For each* $j \in J$, *there exists* $s \in \mathcal{S}$ *such that* $V^s_j(p_s) > 0$, *for all price* $p_s \gg 0$.

(A5) *For each* $h \in \mathcal{H}$, *we assume that:*

  (i) *the correspondence* $\Phi^h : \mathbb{R}^L_+ \twoheadrightarrow \mathbb{R}^J_+$ *has a closed and convex graph.*

  (ii) *for each* $x^h_0 \in \mathbb{R}^L_+$, $0 \in \Phi^h(x^h_0)$ *and* $\Phi^h(x^h_0) \subseteq \Phi^h(x^h_0 + y)$, $\forall y \in \mathbb{R}^L_+$.

(A6) *For each* $j \in \mathcal{J}$, *there is* $h \in \mathcal{H}$ *such that, for any* $x^h_0 \in \mathbb{R}^L_{++}$ *there exists* $\delta_j(x^h_0) > 0$ *such that* $\delta_j(x^h_0)\, e(j) \in \Phi(x^h_0)$, *where* $e(j)$ *denotes the canonical vector of* $\mathbb{R}^J$ *on the* j*-th component.*

(A7) *For each* $h \in \mathcal{H}$, *the correspondence* Φ^h *has compact values.*

Then, our economy has an equilibrium.

Assumptions (A1), (A4) and (A5)(i) are classical. Assumption (A2), is an asymptotic property on preferences which is in particular satisfied by utilities that are time-separable and quasi-linear at $t = 0$. This assumption, jointly with Assumption (A6), allows us to find upper bounds on asset prices (see Lemma 2 in the Appendix). These upper bounds assure that commodity prices can be normalized independently of asset prices, guarantying the lower-hemicontinuity of budget set correspondences (see Lemma 1 in the Appendix). This trick is used to circumvent financial survival assumptions and spanning conditions on admissible portfolio spaces. Assumption (A3)

[3]Given a convex set $X \subset \mathbb{R}^k$, a function $f : X \to \mathbb{R}$ is strongly quasi-convave if $f(\lambda x+(1-\lambda)y) > \min\{f(x), f(y)\}$, for any $(x, y) \in X \times X$ such that $f(x) \neq f(y)$. This property is weaker than strictly quasi-concavity, which requires $f(\lambda x + (1 - \lambda)y) > \min\{f(x), f(y)\}$, for any $(x, y) \in X \times X$ such that $x \neq y$.

assumes that the initial *accumulated* endowment of each agent is positive at each state of nature. For a perishable commodity, it is equivalent to require that initial endowment of that commodity is positive at each state of nature. However, for a durable good, (A3) requires the interiority of individual endowments in that commodity at the first period only. This assumption is needed to guarantee the lower hemicontinuity of the budget correspondences (see Lemma 1). Assumptions (A5)(i) and (A7) allow us to prove that the budget set correspondences have convex and compact values. Assumption (A5)(ii) assumes that credit opportunities do not decrease as purchases of durable goods increase. This is because ownership of durable goods may increase credit opportunities as (depreciated) durable commodities may serve as a partial debt recovery. That is, agents with higher accumulated wealth are more likely to be solvent in the second period and, therefore, have larger debt opportunities. Assumption (A5)(ii) assures that commodity prices are strictly positive, which guarantees market clearing conditions (see Lemma 3 in the Appendix).

3. Applications

Nominal asset markets

Suppose that assets are nominal and that the non-redudancy assumption is satisfied, i.e.: there is a matrix $R \in M_{S\times J}(\mathbb{R})$ such that, for any vector of prices p, $V(p) = R$ with $\operatorname{rank} R = J$. In addition, assume that assumptions (A1)-(A6) hold. In such a case, using monotonicity of preferences and Cramer's rule, we can find endogenous bounds on short-sales.[4] More precisely, there is an endogenous upper bound, $\alpha > 0$, on short sales, i.e.: any budgetary feasible debt satisfies $\varphi^h < \alpha$, for any $h \in \mathcal{H}$. Thus, to prove equilibrium existence, there is no loss of generality to restrict financial participation using the constraint $\overline{\varphi}^h \in \Phi^h(\overline{x}_0^h) \cap [0,\alpha]^J$. By redefining the correspondence of admissible financial positions Φ^h to incorporate the set $[0,\alpha]^J$, we can guarantee that Assumption (A7) also holds. Then, the previous theorem guarantees that equilibrium exists for nominal asset markets with durable goods and endogenous financial participation constraints.

Real asset markets with endogenous short-sales constraints

Under assumptions (A1)-(A6), suppose that assets are real, i.e.: for any $j \in \mathcal{J}$, $V_j^s(p_s) = p_s A_j^s$, where $(A_j^s; s \in \mathcal{S}) \in \mathbb{R}_+^{LS} \setminus \{0\}$. In addition, assume that for any consumption bundle $x_0^h \in \mathbb{R}_+^L$,

$$\Phi^h(x_0^h) \subseteq \{\varphi \in \mathbb{R}_+^J : \varphi \leq m^h(x_0)\},$$

[4]Indeed, using Cramer's Rule, portfolios can be represented by a continuous function of commodity prices at states of nature $s \in \mathcal{S}$ and the associated consumption bundles. Since commodity prices are in the simplex and consumption bundles are non-negative and bounded from above by aggregated endowments, it follows that financial portfolios are also bounded.

where $m^h : \mathbb{R}^L_+ \to \mathbb{R}^J_+$ is a continuous, non-decreasing and concave function. Moreover, suppose that for any asset $j \in \mathcal{J}$, there is some agent $h \in \mathcal{H}$ such that $m^h_j(x^h_0) > 0$ for all $x^h_0 \in \mathbb{R}^L_{++}$. Then, all assumptions of the theorem above hold. We conclude that there exists equilibrium in real financial markets with durable goods, as long as participation constraints assure that short-sales are bounded.

Collateralized asset markets

Given $(s, j) \in \mathcal{S} \times \mathcal{J}$, for any $p_s \in \mathbb{R}^L_+$ let $(A^s_j; s \in \mathcal{S}) \in \mathbb{R}^{LS}_+$ be the vector of real promises of asset $j \in \mathcal{J}$. As in Dubey, Geanakoplos and Zame (1995) and Geanakoplos and Zame (1997, 2002, 2007) we assume that each asset $j \in \mathcal{J}$ is subject to default and backed by physical resources. More precisely, let $C_j \in \mathbb{R}^L_+$ be the bundle of commodities that a borrower of one unit of asset j needs to constitute at the first period as a collateral guarantee. In the absence of any payment enforcement over collateral repossession, asset payments satisfy $V^s_j(p_s) = \min\{p_s A^s_j, p_s Y_s C_j\}$. Assume that, for any $j \in \mathcal{J}$, there is $s \in \mathcal{S}$ such that $\min\{\|A^s_j\|_L, \|Y_s C_j\|_L\} > 0$.[5] In addition, since borrowers are burden to constitute the collateral guarantees, for any $(h, x^h_0) \in \mathcal{H} \times \mathbb{R}^L_+$, we assume that

$$\Phi^h(x^h_0) = \left\{ \varphi^h \in \Omega^h : \sum_{j \in J} C_j \varphi^h_j \leq x^h_0 \right\},$$

where Ω^h is a is a closed and convex subset of $\mathbb{R}^J_+$ containing the vector zero. Also, for any asset $j \in \mathcal{J}$, there is some agent $h \in \mathcal{H}$ such that $\delta e(j) \in \Omega^h$ for some $\delta > 0$.

It follows that assumptions (A4)-(A7) hold and, therefore, if we suppose that preferences and endowments satisfy assumptions (A1)-(A3), then equilibrium exists in Dubey, Geanakoplos and Zame (1995) and Geanakoplos and Zame (1997, 2002, 2007) models of collateralized loans, even when agents have restricted access to some loans.

Note that restricted financial participation is determined by the sets $(\Omega^h; h \in \mathcal{H})$. As we said above, we are particularly interested in the case where Ω^h are positive cones generated by some but not all the canonical vectors of $\mathbb{R}^J$. Indeed, in this context, borrowers may not have access to credit in some assets. This kind of restricted participation is not allowed in models with survival financial assumptions, as these types of hypotheses require that agents have access to all credit markets, independently of prices (see, for instance, Aouani and Cornet (2009, Assumption FN2)).

Appendix

We prove our equilibrium existence result using a generalized game approach. More precisely, given $(n, m) \in \mathbb{N} \times \mathbb{N}$, we define:

$$K(n) = \{(\theta, \varphi) \in \mathbb{R}^J_+ \times \mathbb{R}^J_+ : \varphi_j \leq 2\kappa(n) \ \wedge \ \theta_j \leq 2\kappa(n)J\},$$

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[5]The symbol $\|\cdot\|_L$ denotes the Euclidean norm of $\mathbb{R}^L_+$.

where

$$\kappa(n)=\max_{h\in\mathcal{H}}\max_{x_0\in[0,n]^L}\max_{\varphi\in\Phi^h(x_0)}\sum_{j\in\mathcal{J}}\varphi_j=\max_{h\in\mathcal{H}}\max_{\varphi\in\Phi^h(n,\ldots,n)}\sum_{j\in\mathcal{J}}\varphi_j,$$

where the last equality follows from Assumption (A5)(ii). Note that, Assumption (A7) assures that $\kappa(n)$ is well defined and Assumption (A5)(ii) implies that $\kappa(n)$ is non-decreasing in n. It follows from Assumption (A6) that $\kappa(n)>0$ for any $n>0$. Let $\Delta=\left\{p\in\mathbb{R}^L_+:\sum_{l\in L}p_l=1\right\}$ and $W=\max_{(s,l)\in\mathcal{S}^*\times\mathcal{L}}\sum_{h\in\mathcal{H}}W^h_{s,l}$ be an upper bound for accumulated physical resources in our economy. In addition, let $\mathbb{Y}(n)=[0,n]^L\times[0,2W]^{SL}\times K(n)$.

Consider a game $\mathcal{G}(n,m)$ with $H+S+1$ players. Each agent $h\in\mathcal{H}$ takes as given prices $(p,q)\in\Delta\times[0,m]^J$ and chooses a plan (x^h,θ^h,φ^h) in his truncated budget set $B^h_n(p,q):=B^h(p,q)\cap\mathbb{Y}(n)$ in order to maximize his utility function u^h.

Moreover, there is a player a_0 who takes as given plans $((x^h,\theta^h,\varphi^h);h\in\mathcal{H})\in\mathbb{Y}(n)^H$ and chooses prices $(p_0,q)\in\Delta\times[0,m]^J$ in order to maximize the function

$$p_0\sum_{h\in\mathcal{H}}\left(x^h_0-w^h_0\right)+\sum_{j\in J}q_j\sum_{h\in\mathcal{H}}(\theta^h_j-\varphi^h_j).$$

Finally, for any $s\in S$, there is a player a_s who takes as given plans $((x^h,\theta^h,\varphi^h);h\in\mathcal{H})\in\mathbb{Y}(n)^H$ and chooses prices $p_s\in\Delta$ in order to maximize the function $p_s\sum_{h\in\mathcal{H}}\left(x^h_s-\left(w^h_s+Y_sx^h_0\right)\right)$.

Definition. *A Nash equilibrium for the generalized game $\mathcal{G}(n,m)$ is a vector*

$$\Big((\overline{p},\overline{q});((\overline{x}^h,\overline{\theta}^h,\overline{\varphi}^h);h\in\mathcal{H})\Big)\in\Delta^{S+1}\times[0,m]^J\times\mathbb{Y}(n)^H$$

such that each player maximizes his objective function taking as given the choices of the other players.

Lemma 1. *Under assumptions (A1)-(A7), for each $(n,m)\in\mathbb{N}\times\mathbb{N}$, the game $\mathcal{G}(n,m)$ has a Nash equilibrium.*

Proof. For each $s\in\mathcal{S}^*$, the objective function of player a_s is continuous in all variables and quasi-concave in the own strategy. In addition, the correspondence of admissible strategies for these players, (i.e., the correspondences that associate to plans $((x^h,\theta^h,\varphi^h);h\in\mathcal{H})\in\mathbb{Y}(n)^H$ the admissible prices) are constant with non-empty, convex and compact values. Thus, these correspondences are also continuous.

On the other hand, it follows from Assumption (A1) that the objective function of each player $h\in\mathcal{H}$ is continuous and quasi-concave in the own strategy. The correspondence B^h_n of admissible strategies is upper hemicontinuous with convex values, since it is closed and has non-empty values that are contained in the compact set $\mathbb{Y}(n)$. The lower hemicontinuity of B_n^h follows from assumptions (A3) and (A5)(ii), since it is the closure of the interior truncated budget set correspondence, denoted by $\overset{\circ}{B}{}_n^h$, which is lower-hemicontinuous.[6]

The existence of a Nash equilibrium follows from the fact that: (i) players' objective functions are continuous and quasi-concave in their own strategy, and (ii) correspondences of admissible strategies are continuous with compact, convex and non-empty values. More precisely, applying Kakutani's fixed point theorem to the product of best response correspondences, we get a Nash equilibrium as a fixed point. $\square$

Lemma 2. *Let $\Big((\overline{p},\overline{q});((\overline{x}^h,\overline{\theta}^h,\overline{\varphi}^h);h\in\mathcal{H})\Big)$ be a Nash equilibrium of the game $\mathcal{G}(n,m)$. Under assumptions (A1)-(A7), if the consumption bundle $\overline{x}_0^h \leq W(1,\dots,1)$ for all $h\in\mathcal{H}$, then for n large enough there exists $\overline{m}\in\mathbb{N}$ such that $\overline{q}_j<\overline{m}$ for each $j\in J$.*

Proof. Given $a>0$, let

$$(\tilde{x}_0^h(a),(\tilde{x}_s^h;s\in\mathcal{S}))=((a,\dots,a)+w_0^h,(0.5\,W_s^h;s\in\mathcal{S}^*))\in\mathbb{R}_{++}^{L(S+1)}$$

Then, it follows from assumptions (A1) and (A2) that there exists $\overline{a}>0$ such that, for each $h\in\mathcal{H}$,

$$u^h\left((\tilde{x}_0(\overline{a}),(\tilde{x}_s^h;s\in\mathcal{S})\right)>u^h\Big(W(1,\dots,1),(2W(1,\dots,1);s\in\mathcal{S})\Big)\geq u^h(\overline{x}_0^h,(\overline{x}_s^h;s\in\mathcal{S})).$$

Thus, under the hypotheses of the lemma, the bundle $(\tilde{x}_0^h(\overline{a}),(\tilde{x}_s^h;s\in\mathcal{S}))$ can not be demanded by any agent at prices $(\overline{p},\overline{q})$.

On the other hand, Assumption (A6) assures that, given an asset $j\in\mathcal{J}$, there exists an agent $h(j)\in\mathcal{H}$ such that, for some $\delta_j^{h(j)}:=\delta_j(\tilde{x}_0^{h(j)}(\overline{a}))>0$, we have $\delta_j^{h(j)}\,e(j)\in\Phi^{h(j)}(\tilde{x}_0^{h(j)}(\overline{a}))$. Suppose that $n>W+\overline{a}$, and that agent $h(j)$ chooses the portfolio $(\tilde{\theta}^h,\tilde{\varphi}^h)=(0,\eta^{h(j)}\,e(j))$, where

[6]The correspondence $\overset{\circ}{B}{}_n^h\colon \Delta^{S+1}\times[0,m]^J\twoheadrightarrow\mathbb{Y}(n)$ associates to each (p,q) the allocations in $B_n^h(p,q)$ that satisfy state contingent budget constraints as strict inequalities. This correspondence has non-empty values, since the consumption bundle $(0.5w_0^h,(0.25W_s^h;s\in\mathcal{S}))$ jointly with the zero financial portfolio always belongs to $\overset{\circ}{B}{}_n^h(p,q)$, independently of the vector of prices $(p,q)\in\Delta^{S+1}\times[0,m]^J$. Also, given any price $(p,q)\in\Delta^{S+1}\times[0,m]^J$ and a sequence $((p_k,q_k);k\in\mathbb{N})\subset\Delta^{S+1}\times[0,m]^J$ that converges to (p,q), for any $(x^h,\theta^h,\varphi^h)\in\overset{\circ}{B}{}_n^h(p,q)$ there exists $N\in\mathbb{N}$ such that $(x^h,\theta^h,\varphi^h)\in\overset{\circ}{B}{}_n^h(p_k,q_k)$ for any $k\geq N$. Then, it follows from the sequential characterization of lower-hemicontinuity that $\overset{\circ}{B}{}_n^h$ is a lower-hemicontinuous correspondence.

Given any $(x^h,\theta^h,\varphi^h)\in B_n^h(p,q)$ and $\lambda\in(0,1)$, one has $\left(\left(\lambda x_0^h+(1-\lambda)\frac{\omega_0^h}{2},(\lambda x_s^h;s\in\mathcal{S})\right),\lambda\theta^h,\lambda\varphi^h\right)\in\overset{\circ}{B}{}_n^h(p,q)$ (since Assumption (A5)(i) assures that Φ^h has a convex graph and $0\in\Phi^h(0)$). Thus, taking the limit as λ goes to zero, we show that (x^h,θ^h,φ^h) belongs to the closure of $\overset{\circ}{B}{}_n^h(p,q)$. Thus, as $\overset{\circ}{B}{}_n^h(p,q)\subseteq B_n^h(p,q)$, it follows that B^h is the closure of the interior truncated budget set correspondence.

$\eta^{h(j)} \in (0, \min\{\kappa(1), \delta_j^{h(j)}\})$ satisfies $\left(\max_{(p,s)\in\Delta\times\mathcal{S}} V_j^s(p)\right)\eta^{h(j)} < 0.25 \min_{(s,l)\in\mathcal{S}\times\mathcal{L}} W_{s,l}^{h(j)}$.[7] Since he can not consume the bundle $(\tilde{x}_0^h(\overline{a}), (\tilde{x}_s^h; s \in \mathcal{S}))$, it follows that $\eta^{h(j)}\overline{q}_j < \overline{a}$, which assures the existence of an upper bound for asset j unitary price (this bound only depends on primitives of the economy). Then, choosing $\overline{m} = \overline{a}\max_{j\in J}\left(\eta^{h(j)}\right)^{-1}$, we conclude the proof. □

Let $n^* = W + \overline{a}$.

LEMMA 3. *Under assumptions (A1)-(A7), and for $n > n^*$, a Nash equilibrium of $\mathcal{G}(n, \overline{m})$ is an equilibrium for our economy.*

PROOF. Let $\left((\overline{p}, \overline{q}); ((\overline{x}^h, \overline{\theta}^h, \overline{\varphi}^h); h \in \mathcal{H})\right) \in \Delta^{S+1} \times [0, \overline{m}]^J \times \mathbb{Y}(n)^H$ be a Nash equilibrium of the generalized game $\mathcal{G}(n, \overline{m})$. Adding first period budget constraints of agents $h \in \mathcal{H}$ we get:

$$\overline{p}_0 \sum_{h\in\mathcal{H}} \left(\overline{x}_0^h - w_0^h\right) + \sum_{j\in J} \overline{q}_j \sum_{h\in\mathcal{H}} (\overline{\theta}_j^h - \overline{\varphi}_j^h) \leq 0.$$

It follows that the optimal value of the objective function of player a_0 is less than or equal to zero and, therefore, for each $l \in \mathcal{L}$, $\sum_{h\in\mathcal{H}} (\overline{x}_{0,l}^h - w_{0,l}^h) \leq 0$. Indeed, otherwise, player a_0 would choose a price equal to one for commodity $l \in \mathcal{L}$ and a zero price for the other commodities and assets, obtaining a positive value for his objective function, a contradiction with the definition of Nash equilibrium. Therefore, for each $h \in \mathcal{H}$, $\overline{x}_0^h \leq W(1, \ldots, 1)$. On the other hand, if for some $j \in \mathcal{J}$, $\sum_{h\in\mathcal{H}} \left(\overline{\theta}_j^h - \overline{\varphi}_j^h\right) > 0$, then $\overline{q}_j = \overline{m}$, which contradicts Lemma 2 for $n > n^*$. Thus, $\sum_{h\in\mathcal{H}} \left(\overline{\theta}^h - \overline{\varphi}^h\right) \leq 0$.

Since $\overline{x}_{0,l}^h < n$ for all $(h, l) \in \mathcal{H} \times \mathcal{L}$, it follows from Assumption (A5)(ii) that first period budget constraints are saturated. Therefore, $\sum_{h\in\mathcal{H}} (\overline{x}_0^h - w_0^h) = 0$. In fact, otherwise, some commodity at $t = 0$ has a zero price, a contradiction with the existence of an interior optimal plan under Assumption (A1). Analogously, if $\sum_{h\in\mathcal{H}} \left(\overline{\theta}_j^h - \overline{\varphi}_j^h\right) < 0$, then $\overline{q}_j = 0$.

Summing up the budget constraints at state of nature $s \in S$ of all agents $h \in \mathcal{H}$, we obtain that,

$$\overline{p}_s \sum_{h\in\mathcal{H}} \left(\overline{x}_s^h - \left(w_s^h + Y_s \overline{x}_0^h\right)\right) \leq 0.$$

Then, the optimal value of player a_s's objective function is less than or equal to zero. This implies that $\sum_{h\in\mathcal{H}} \left(\overline{x}_s^h - \left(w_s^h + Y_s \overline{x}_0^h\right)\right) \leq 0$ and, therefore, $\overline{x}_s^h < 2W(1, \ldots, 1)$. Thus, $\overline{p}_s \gg 0$, which assures together with Assumption (A4) that $\overline{q} \gg 0$, guarantying financial market feasibility. Moreover, it follows from Assumption (A1) that second period budget constraints are satisfied as equalities.

[7] To make $\eta^{h(j)}$ a feasible debt for agent $h(j)$ in the game $\mathcal{G}(n, m)$, i.e. $\eta^{h(j)} < \kappa(n)$, it is sufficient to assure that $\eta^{h(j)} < \kappa(1)$ (an upper bound that depends only on primitives), as $(\kappa(n); n \in \mathbb{N})$ is a non-decreasing and strictly positive sequence of n.

Then, $\overline{p}_s \sum_{h\in\mathcal{H}} \left(\overline{x}_s^h - \left(w_s^h + Y_s \overline{x}_0^h\right)\right) = 0$. Since $\overline{p}_s \gg 0$, we conclude that $\sum_{h\in\mathcal{H}} \left(\overline{x}_s^h - \left(w_s^h + Y_s \overline{x}_0^h\right)\right) = 0$. It follows that market clearing conditions of equilibrium definition are satisfied.

On the other hand, for each agent $h \in H$, the plan $(\overline{x}^h, \overline{\theta}^h, \overline{\varphi}^h) \in B_n^h(\overline{p}, \overline{q}) \subset B^h(\overline{p}, \overline{q})$ belongs to $\text{int}(K(n))$ (relative to $\mathbb{R}_+^L \times \mathbb{R}_+^J \times \mathbb{R}_+^J$). Therefore, the strong quasi-concavity of u^h, jointly with the convexity of budget sets, implies that $(\overline{x}^h, \overline{\theta}^h, \overline{\varphi}^h)$ is also optimal in $B^h(\overline{p}, \overline{q})$. □

References

[1] Angeloni, L. and B. Cornet (2006): "Existence of financial equilibria in a multi-period stochastic economy," *Advances in Mathematical Economics*, 8, 933-955.

[2] Aouani, Z. and B. Cornet (2009): "Existence of financial equilibria with restricted participation," *Journal of Mathematical Economics*, 45, 772-786.

[3] Carosi, L., M. Gori, and A. Villanacci (2009): "Endogenous restricted participation in general financial equilibrium," *Journal of Mathematical Economics*, 36, 61-76.

[4] Cass, D, P. Siconolfi and A. Villanacci (2001): "Generic regularity of competitive equilibria with restricted participation," *Journal of Mathematical Economics*, 45, 787-806.

[5] Cornet, B., and R. Gopalan (2010): "Arbitrage and equilibrium with portfolio constraints ," *Economic Theory*, doi: 10.1007/s00199-009-0506-5.

[6] Dubey, P., J. Geanakoplos, and W.R. Zame (1995):"Collateral, default and market crashes," Discussion paper, Cowles Foundation, Yale University.

[7] Geanakoplos, J., and W.R. Zame (1997):"Collateral, default and market crashes," Discussion paper, Cowles Foundation, Yale University.

[8] Geanakoplos, J., and W.R. Zame (2002):"Collateral and the Enforcement of Intertemporal Contracts," Discussion paper, Cowles Foundation, Yale University.

[9] Geanakoplos, J., and W.R. Zame (2007):"Collateralized asset markets," Discussion paper, University of California Los Angeles.

Economics Department, School of Business, American University in Cairo
AUC Avenue, P.O. Box 74 New Cairo 11835, Egypt.
E-mail address: `kseghir@aucegypt.edu`

Departament of Economics,University of Chile
Diagonal Paraguay 257, Santiago, Chile
E-mail address: `juan.torres@fen.uchile.cl`